Al-Ayen University

College of Petroleum
Engineering

Numerical Methods and Reservoir Simulation

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Lecture 5: Principles of Finite Difference Approximation
(Finite Difference Scheme & Discretization)

Finite Differences

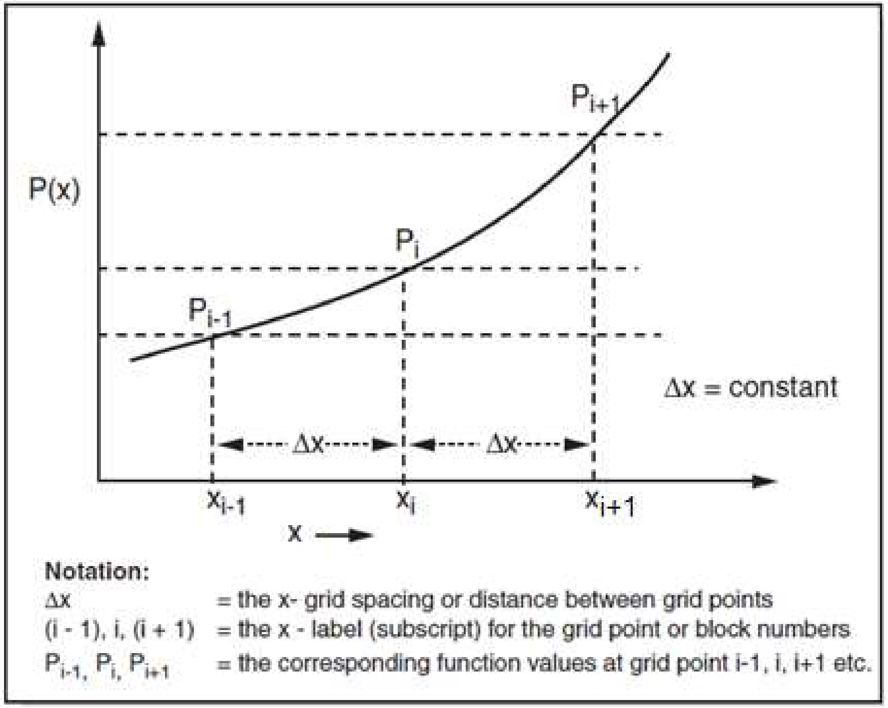
Definition: The finite difference scheme is a way of approximating derivatives of a function. There are three ways to do that :

Approximation 1 **-Forward Differences (*fd*):**

$$\left(\frac{dP}{dx}\right)_{i\,fd} = \frac{P_{i+1} - P_i}{\Delta x}$$

Approximation 2 **-Backward Differences (*bd*):**

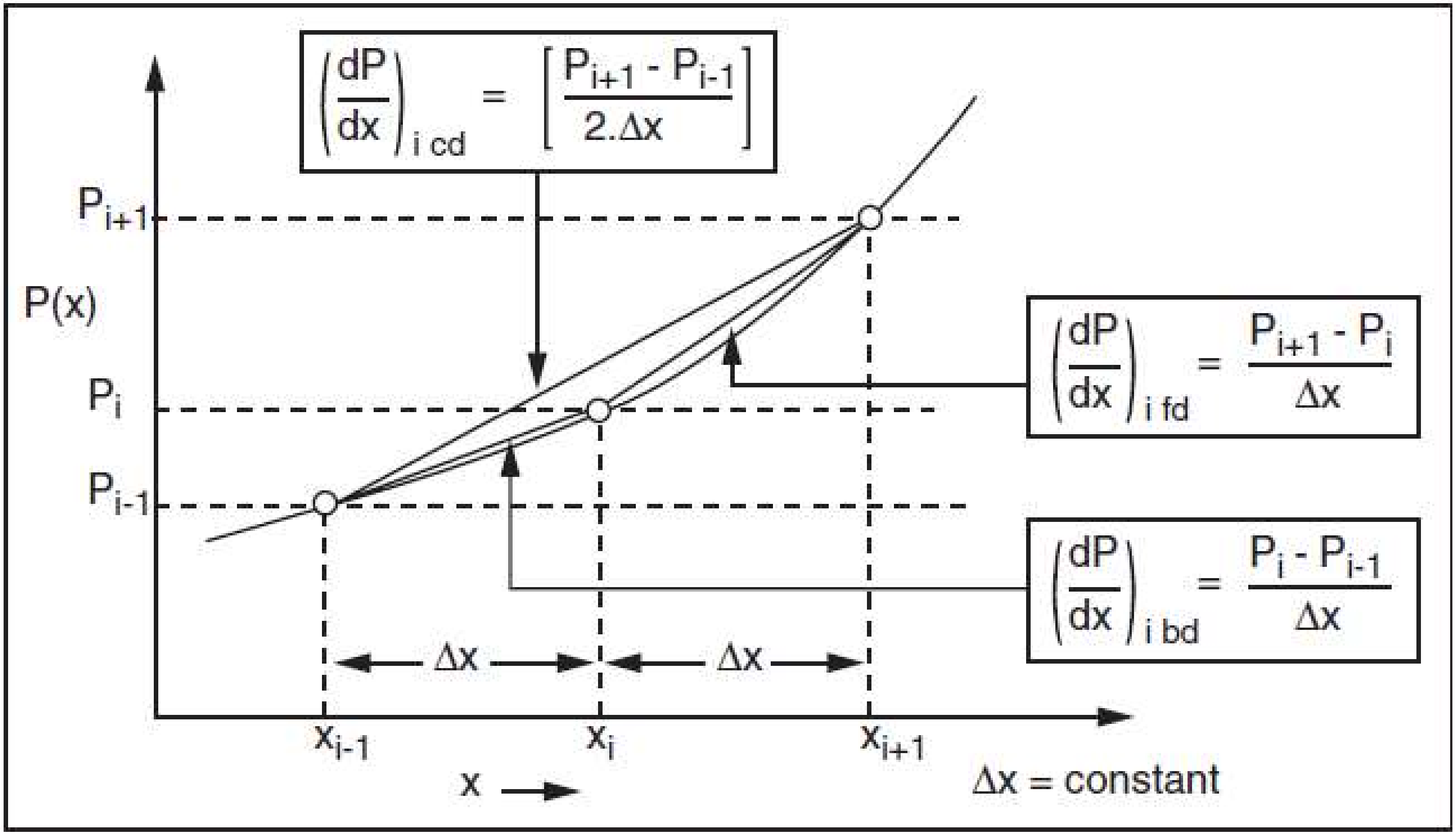
$$\left(\frac{dP}{dx}\right)_{i\,bd} = \frac{P_i - P_{i-1}}{\Delta x}$$



Approximation 3 **-Central Differences (*cd*):**

$$\left(\frac{dP}{dx}\right)_{i\,cd} = \frac{1}{2}\left[\left(\frac{dP}{dx}\right)_{i\,fd} + \left(\frac{dP}{dx}\right)_{i\,bd}\right] = \frac{1}{2}\left[\frac{P_{i+1} - P_i}{\Delta x} + \frac{P_i - P_{i-1}}{\Delta x}\right]$$

$$\left(\frac{dP}{dx}\right)_{i\,cd} = \left[\frac{P_{i+1} - P_{i-1}}{2.\Delta x}\right]$$



Graphical illustration of the finite difference derivatives calculated by backward differences (*bd*), forward differences (*fd*) and central differences(*cd*) .

- ***Now consider the finite difference approximation of the second derivative*:**

$$\left(\frac{\partial^2 P}{\partial x^2}\right) \approx \frac{\left(\frac{dP}{dx}\right)_{i\,fd} - \left(\frac{dP}{dx}\right)_{i\,bd}}{\Delta x}$$

$$\approx \frac{\left(\frac{P_{i+1} - P_i}{\Delta x}\right)_{i\,fd} - \left(\frac{P_i - P_{i-1}}{\Delta x}\right)_{i\,bd}}{\Delta x}$$

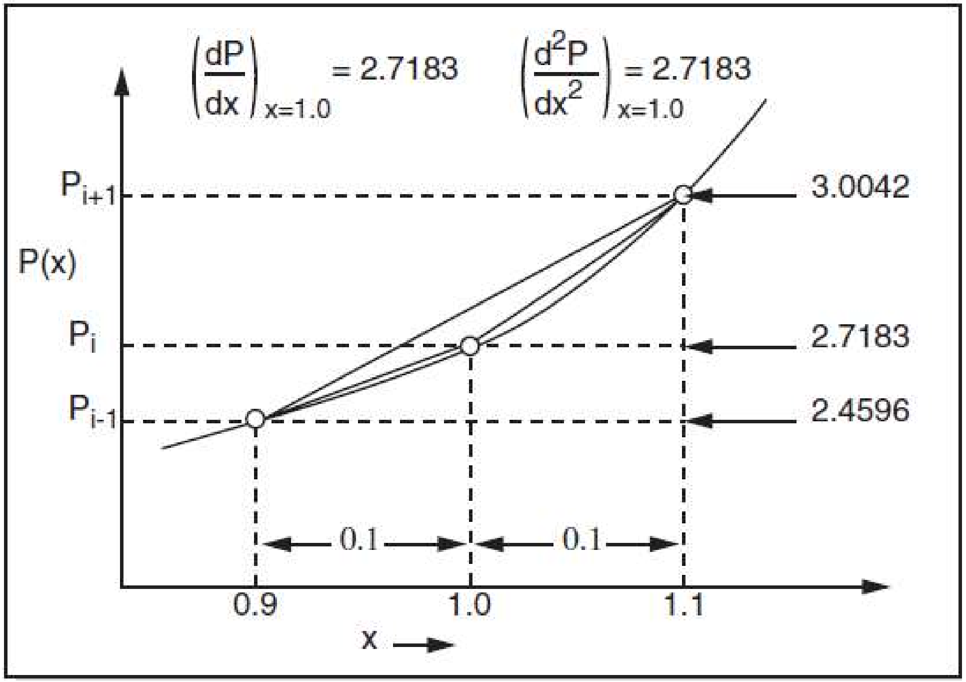
$$\left(\frac{\partial^2 P}{\partial x^2}\right) \approx \frac{\left(P_{i+1} + P_{i-1} - 2P_i\right)}{\Delta x^2}$$

Numerical Example

$$\left(\frac{dP}{dx}\right)_{i\,fd} \approx \left[\frac{3.0042-2.7183}{0.1}\right] = 2.859 \;\; [err \approx +0.14]$$

$$\left(\frac{dP}{dx}\right)_{i\,bd} \approx \left[\frac{2.7183-2.4596}{0.1}\right] = 2.587 \;\; [err \approx -0.13]$$

$$\left(\frac{dP}{dx}\right)_{i\,cd} \approx \left[\frac{3.0042-2.4596}{2x0.1}\right] = 2.723 \;\; [err \approx -0.005]$$



A numerical example where the function values and derivative values are known ($P(x) = e^x$)

$$\left(\frac{\partial^2 P}{\partial x^2}\right) \approx \frac{3.0042+2.4596-2x2.7183}{0.1^2} = 2.7200 \; [err \approx 0.0017]$$

Identifying Error of Finite Difference Using Taylor Series

- A so-called Taylor series approximation of a function $f(x+h)$ expressed in terms of $f(x)$ and its derivatives $f'(x)$ may be written:

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots.$$

- Applying Taylor series to our pressure function.

Approximation of the second order space derivative

At constant time, t, the pressure function may be expanded forward and backwards:

$$P(x+\Delta x, t) = P(x,t) + \frac{\Delta x}{1!} P'(x,t) + \frac{(\Delta x)^2}{2!} P''(x,t) + \frac{(\Delta x)^3}{3!} P'''(x,t) + \ldots.$$

$$P(x-\Delta x, t) = P(x,t) + \frac{(-\Delta x)}{1!} P'(x,t) + \frac{(-\Delta x)^2}{2!} P''(x,t) + \frac{(-\Delta x)^3}{3!} P'''(x,t) + \ldots.$$

By adding these two expressions, and solving for the second derivative, we get the following approximation:

$$P''(x,t) = \frac{P(x+\Delta x, t) - 2P(x,t) + P(x+\Delta x, t)}{(\Delta x)^2} + \frac{(\Delta x)^2}{12} P''''(x,t) + \ldots.$$

$$P''(x,t) = \frac{P(x+\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)}{(\Delta x)^2} + \frac{(\Delta x)^2}{12} P''''(x,t) + \ldots.$$

or, by employing the grid index system, and using superscript to indicate time level:

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i^t = \frac{P_{i+1}^t - 2P_i^t + P_{i-1}^t}{(\Delta x)^2} + O(\Delta x^2)$$

- Here, the rest of the terms from the Taylor series expansion are collectively denoted $O(\Delta x^2)$ thus denoting that they are in order of, or proportional in size to Δx^2
- This error term is called ***discretization error (or truncation error)*** , which in this case is of second order, is neglected in the numerical solution.
- The smaller the grid blocks used, the smaller will be the error involved.
- Any time level could be used in the expansions above.

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{t+\Delta t} = \frac{P_{i+1}^{t+\Delta t} - 2P_i^{t+\Delta t} + P_{i-1}^{t+\Delta t}}{(\Delta x)^2} + O(\Delta x^2)$$

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{t+\frac{\Delta t}{2}} = \frac{P_{i+1}^{t+\frac{\Delta t}{2}} - 2P_i^{t+\frac{\Delta t}{2}} + P_{i-1}^{t+\frac{\Delta t}{2}}}{(\Delta x)^2} + O(\Delta x^2)$$

Approximation of the time derivative

At constant position, x, the pressure function may be expanded in forward direction in regard to time:

$$P(x, t+\Delta t) = P(x, t) + \frac{\Delta t}{1!} P'(x, t) + \frac{(\Delta t)^2}{2!} P''(x, t) + \frac{(\Delta t)^3}{3!} P'''(x, t) + \ldots..$$

By solving for the first derivative, we get the following approximation:

$$P'(x,t) = \frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} - \frac{(\Delta t)}{2} P''(x,t) - \ldots..$$

or, employing the index system:

$$\left(\frac{\partial P}{\partial t}\right)_i^t = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t)$$

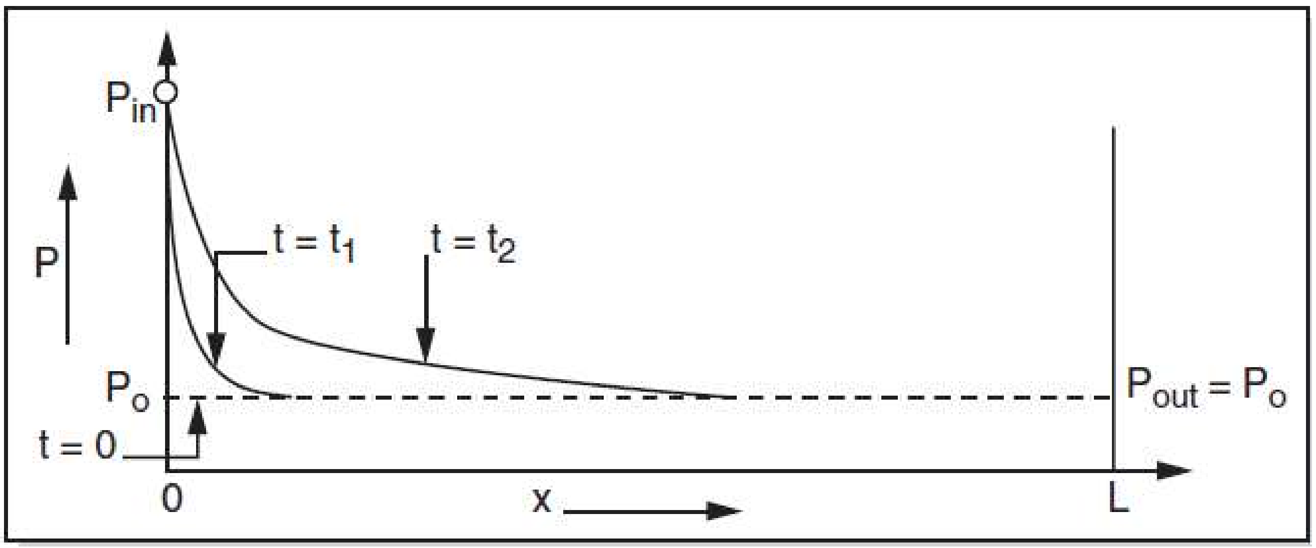
- Here, the error term is proportional to Δt, or of the first order.
- The error therefore approaches zero slower in this case than for the second order term $O(\Delta x^2)$.

Application of Finite Differences to Partial Differential Equations (PDEs)

As an example of PDE, we will take the simplified pressure equation:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} \qquad \text{------------- (1), where} \qquad \eta = \frac{k}{\phi c_t \mu}$$

This is the pressure equation for a 1D system where $0 \le x \le L$, where L is the length of the system. After the system is held constant at P = Po, the inlet pressure is raised (at x = 0) instantly to P = Pin while the outlet pressure is held at Pout = Po.



Explicit Finite Difference Approximation of the Linear Pressure Equation

PDE:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} \qquad \text{------------- (1)}$$

- apply finite differences to equation 1 to obtain:

$$\left(\frac{\partial P}{\partial t}\right)_i \approx \frac{P_i^{n+1} - P_i^n}{\Delta t} \qquad \text{------------- (2)}$$

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i \approx \frac{P_{i+1}^n + P_{i-1}^n - 2P_i^n}{\Delta x^2} \qquad \text{------------- (3)}$$

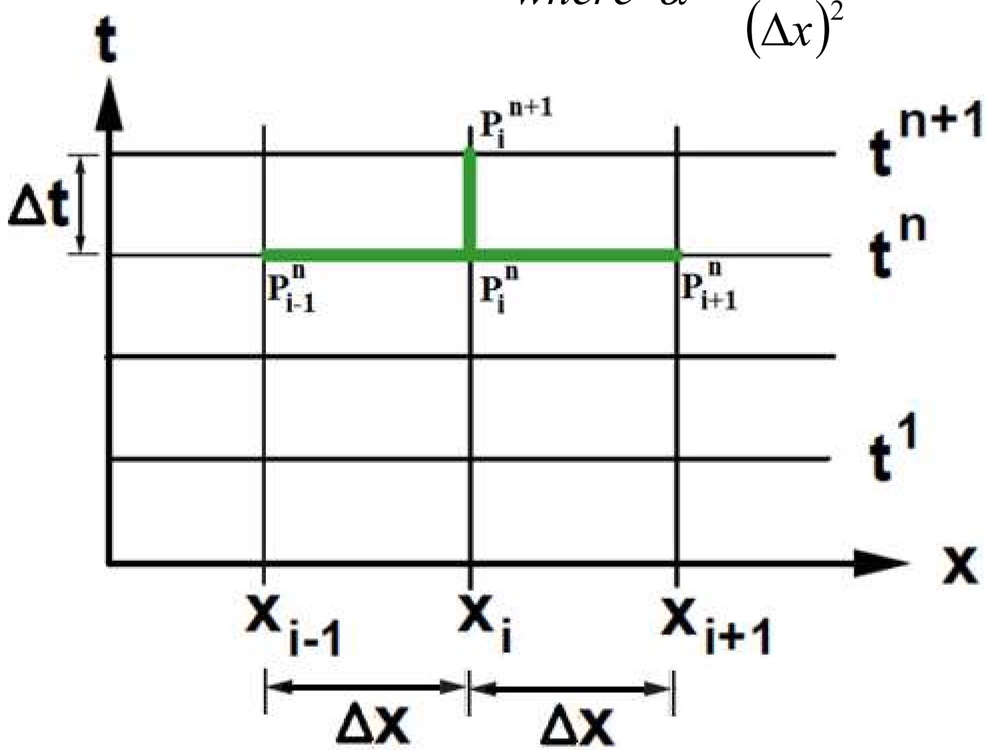
Substituting equations 2 and 3 in equation 1 gives:

$$\textit{Explicit Finite Difference} \quad \frac{p_{i+1}^{n} - 2p_i^n + p_{i-1}^n}{(\Delta x)^2} = \frac{1}{\eta}\left(\frac{p_i^{n+1} - p_i^n}{\Delta t}\right) \quad \text{------------- (4)}$$

Rearrange equation 4 to obtain an ***explicit expression*** for, p_i^{n+1} , the only unknown in equation 4:

$$p_i^{n+1} = \alpha\, p_{i-1}^n + (1-2\alpha)p_i^n + \alpha\, p_{i+1}^n \quad \text{------------- (5)}$$

$$where \;\; \alpha = \frac{\eta \Delta t}{(\Delta x)^2}$$



Schematic of the explicit finite difference algorithm for solving the simple PDE.

Implicit Finite Difference Approximation of the Linear Pressure Equation

We now return to the original PDE

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} \qquad \text{------------- (1)}$$

The finite difference equation for this case is as follows:

The time derivative is the same as for the explicit finite difference , i.e.

$$\left(\frac{\partial P}{\partial t}\right)_i \approx \frac{P_i^{n+1} - P_i^n}{\Delta t} \qquad \text{------------- (2)}$$

but the spatial derivative of equation 3 (Part 1) now becomes:

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i \approx \frac{P_{i+1}^{n+1} + P_{i-1}^{n+1} - 2P_i^{n+1}}{\Delta x^2} \qquad \text{------------- (6)}$$

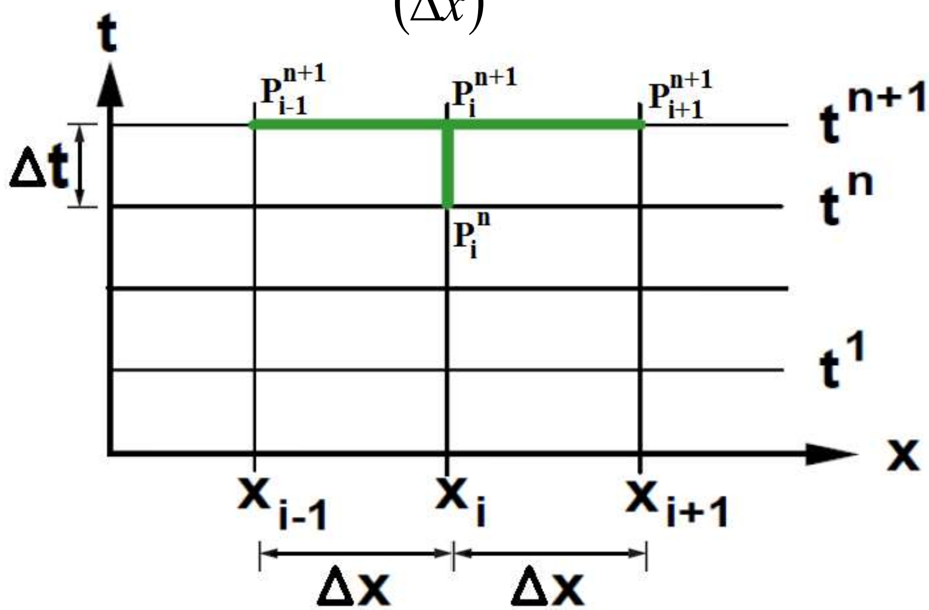
Substituting equations 2 and 6 in equation 1 gives:

$$\textit{Implicit Finite Difference}\quad \frac{p_{i+1}^{n+1}-2p_i^{n+1}+p_{i-1}^{n+1}}{(\Delta x)^2}=\frac{1}{\eta}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t}\right) \quad \text{------------- (7)}$$

There are three unknowns for each i; p_{i-1}^{n+1}, p_i^{n+1}, and p_{i+1}^{n+1}. Equation 7 can be rearranged as:

$$-\alpha\, p_{i-1}^{n+1}+(1+2\alpha)\,p_i^{n+1}-\alpha\, p_{i+1}^{n+1}=p_i^n \quad \text{------------- (8)}$$

$$\textit{where}\ \ \alpha=\frac{\eta\Delta t}{(\Delta x)^2}$$



Schematic of the implicit finite difference algorithm for solving the simple pressure PDE

Discretization

- Converts continuous PDEs into difference form
- Replaces original problem with other problem in terms of algebraic equation which can be solved “easily”
- The reservoir (spatial) domain is represented by spatially distributed, interconnected discrete elements (***grid blocks***)
- Temporal (time) domain is also discretized (***time steps***).

THANK YOU