Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

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2020/2021

L17: Application of Finite Difference Approximation for 2D and 3D Systems (Part 1)

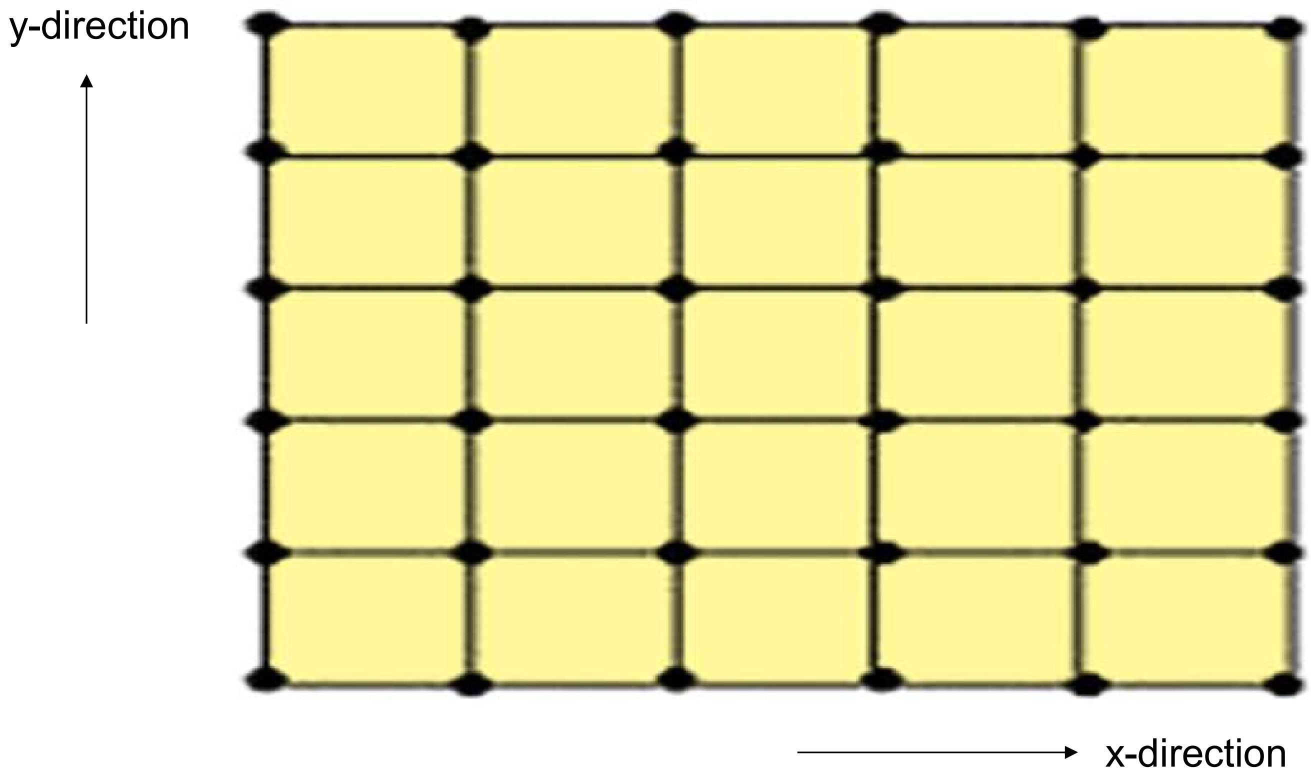
Outline

- 2D and 3D Systems
- 2D Cartesian System/Point-Centered
- 2D Cartesian System/Block-Centered
- 3Dimensional-Cartesian
- 3D (Cylindrical)
- 2D Single-Phase Flow Problem

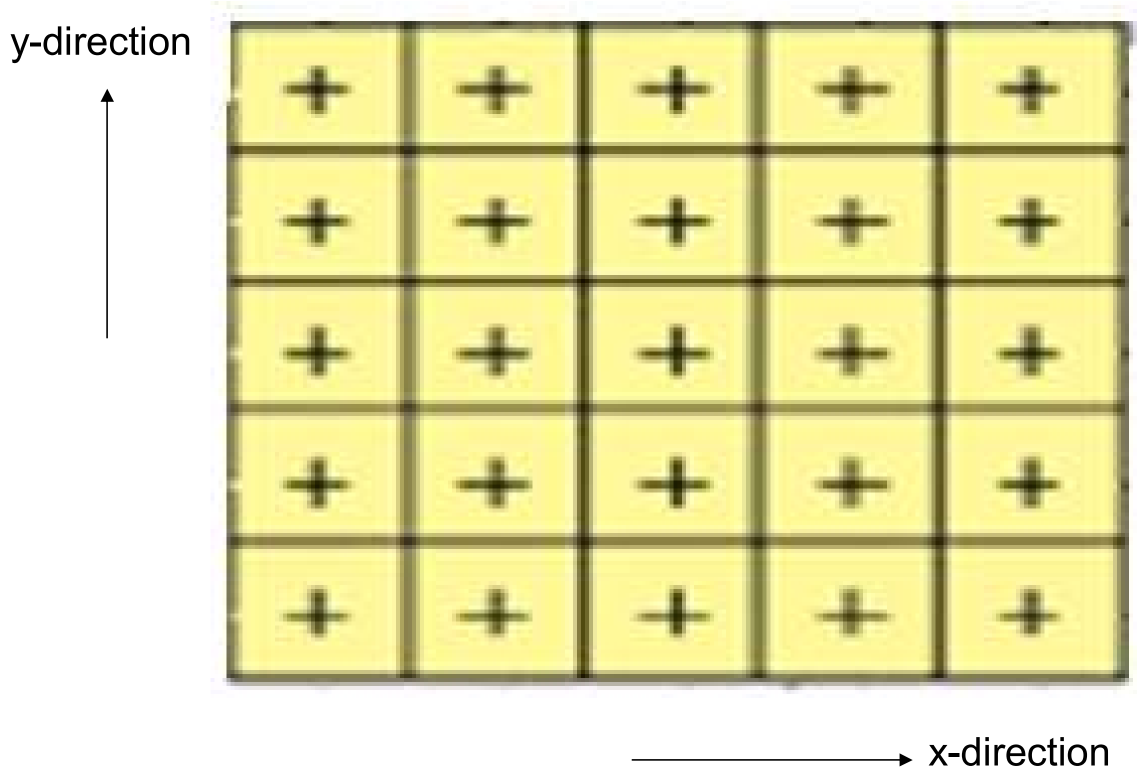
2D and 3D Systems

- Extension application of finite difference approximation to 2D and 3D Cartesian systems are straight forward.
  - For 2D x-y system, we use the indices i and j to show the x- and y-directions, respectively (e.g., $p_{i,j}^{n}$).
  - For 3D x-y-z system, we use the indices *i*, *j*, and *k* to show the *x*-, *y*-, and *z*-directions, respectively, such $p_{i,j,k}^{n}$.

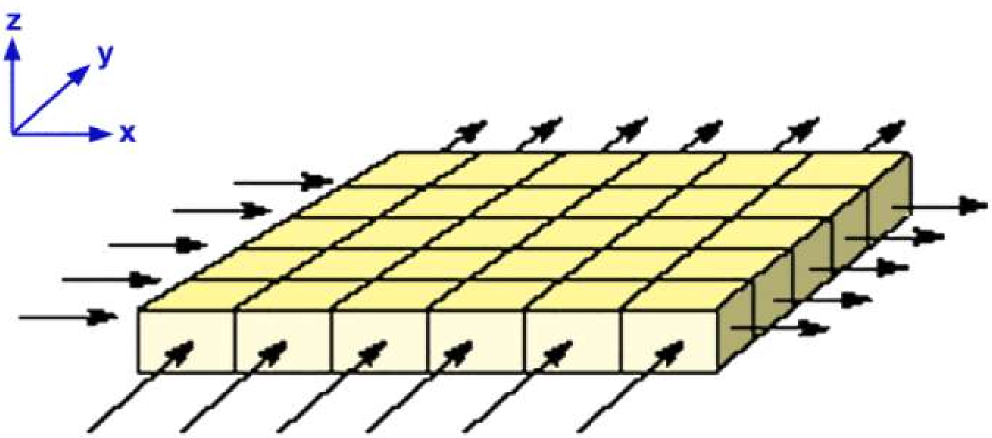
2D Cartesian System/Point-Centered



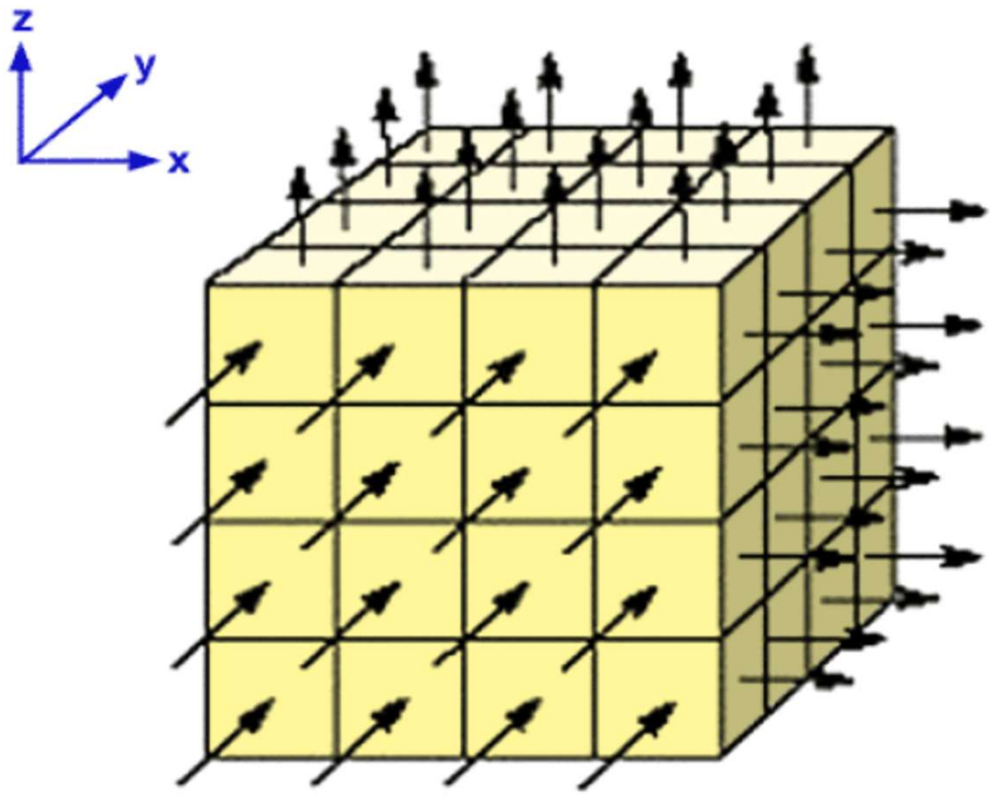
2D Cartesian System/Block-Centered



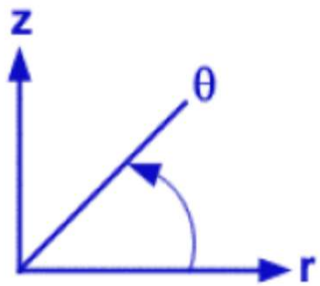
2Dimensional (X-Y) Block-Centered

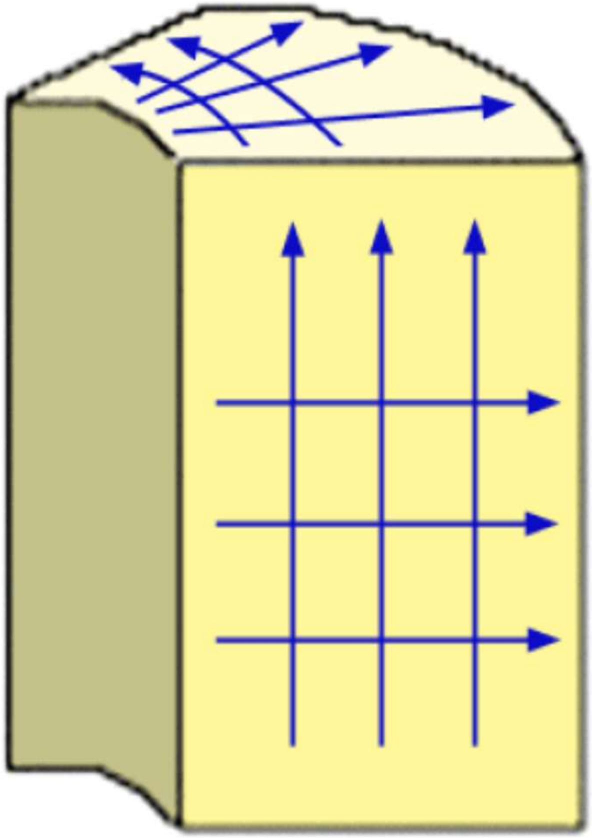


3Dimensional-Cartesian



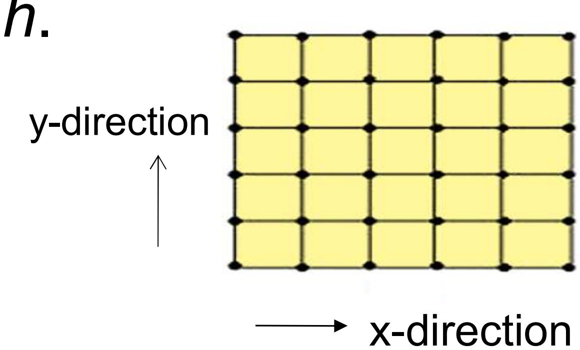
3D (Cylindrical)

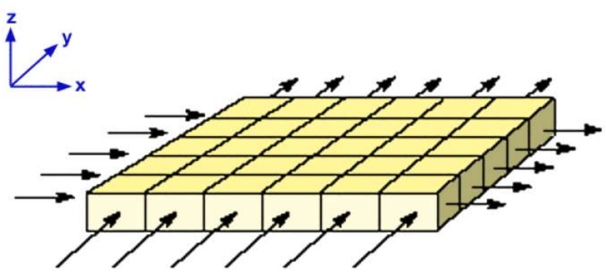




2D Single-Phase Flow Problem

- Let's consider 2-D *x-y* flow of slightly compressible fluid in a reservoir uniform thickness *h*.





$$1.127\times10^{-3}\left[\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k_y}{\mu}\frac{\partial p}{\partial y}\right)\right]-\frac{q_{sc}(x,y,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},$$

$$0<x<L_x, 0<y<L_y, t>0$$

2D Single-Phase Flow Problem

- In this case, we use *i,j* indexing to denote *x* and *y* directions, respectively, and difference partial derivatives in each direction as we did in the 1-D x case. Let's consider implicit method, and hence evaluate the PDE at (x_i, y_j) at t^{n+1}.

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial p}{\partial x}\right)_{x_i, y_j, t^{n+1}} = \frac{\lambda_{x,i+1/2,j}\left(\frac{\partial p}{\partial x}\right)_{i+1/2,j}^{n+1} - \lambda_{x,i-1/2,j}\left(\frac{\partial p}{\partial x}\right)_{i-1/2,j}^{n+1}}{x_{i+1/2} - x_{i-1/2}} \quad \text{where } \lambda_x = k_x / \mu$$

$$\frac{\partial}{\partial y}\left(\lambda_y \frac{\partial p}{\partial y}\right)_{x_i, y_j, t^{n+1}} = \frac{\lambda_{y,i,j+1/2}\left(\frac{\partial p}{\partial y}\right)_{i,j+1/2}^{n+1} - \lambda_{y,i,j-1/2}\left(\frac{\partial p}{\partial y}\right)_{i,j-1/2}^{n+1}}{y_{j+1/2} - y_{j-1/2}} \quad \text{where } \lambda_y = k_y / \mu$$

2D Single-Phase Flow Problem

$$\left(\frac{\partial p}{\partial x}\right)^{n+1}_{i+1/2,j} = \frac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{x_{i+1} - x_i} \qquad \left(\frac{\partial p}{\partial x}\right)^{n+1}_{i-1/2,j} = \frac{p^{n+1}_{i,j} - p^{n+1}_{i-1,j}}{x_i - x_{i-1}}$$

$$\left(\frac{\partial p}{\partial y}\right)^{n+1}_{i,j+1/2} = \frac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{y_{j+1} - y_j} \qquad \left(\frac{\partial p}{\partial y}\right)^{n+1}_{i,j-1/2} = \frac{p^{n+1}_{i,j} - p^{n+1}_{i,j-1}}{y_j - y_{j-1}}$$

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial p}{\partial x}\right)_{x_i, y_j, t^{n+1}} = \frac{\lambda_{x,i+1/2,j}\left(\dfrac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{x_{i+1} - x_i}\right) - \lambda_{x,i-1/2,j}\left(\dfrac{p^{n+1}_{i,j} - p^{n+1}_{i-1,j}}{x_i - x_{i-1}}\right)}{x_{i+1/2} - x_{i-1/2}}$$

$$\frac{\partial}{\partial y}\left(\lambda_y \frac{\partial p}{\partial y}\right)_{x_i, y_j, t^{n+1}} = \frac{\lambda_{y,i,j+1/2}\left(\dfrac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{y_{j+1} - y_j}\right) - \lambda_{y,i,j-1/2}\left(\dfrac{p^{n+1}_{i,j} - p^{n+1}_{i,j-1}}{y_j - y_{j-1}}\right)}{y_{j+1/2} - y_{j-1/2}}$$

2D Single-Phase Flow Problem

$$\left(\frac{\partial p}{\partial t}\right)_{x_i, y_j, t^{n+1}} = \left(\frac{p_{i,j}^{n+1} - p_{i,j}^{n}}{\Delta t^{n+1}}\right), \ \textit{where} \ \ \Delta t^{n+1} = t^{n+1} - t^{n}$$

$$1.127\times10^{-3}\left[\frac{\lambda_{x,i+1/2,j}\left(\frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{x_{i+1} - x_i}\right) - \lambda_{x,i-1/2,j}\left(\frac{p_{i,j}^{n+1} - p_{i-1,j}^{n+1}}{x_i - x_{i-1}}\right)}{x_{i+1/2} - x_{i-1/2}} + \frac{\lambda_{y,i,j+1/2}\left(\frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{y_{j+1} - y_j}\right) - \lambda_{y,i,j-1/2}\left(\frac{p_{i,j}^{n+1} - p_{i,j-1}^{n+1}}{y_j - y_{j-1}}\right)}{y_{j+1/2} - y_{j-1/2}}\right] - \frac{q_{sc,i,j}^{\,n+1} B}{\Delta x_i \Delta y_j h}$$

$$= \frac{(\phi c_t)_{i,j}}{5.615}\left(\frac{p_{i,j}^{n+1} - p_{i,j}^{n}}{\Delta t^{n+1}}\right)$$

2D Single-Phase Flow Problem

- Multiply both sides by the bulk volume of the block, $V_b = \Delta x_i \Delta y_j h$

$$1.127\times10^{-3}\Delta x_i\Delta y_j h\left[\frac{\lambda_{x,i+1/2,j}\left(\dfrac{p_{i+1,j}^{n+1}-p_{i,j}^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2,j}\left(\dfrac{p_{i,j}^{n+1}-p_{i-1,j}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}+\frac{\lambda_{y,i,j+1/2}\left(\dfrac{p_{i+1,j}^{n+1}-p_{i,j}^{n+1}}{y_{j+1}-y_j}\right)-\lambda_{y,i,j-1/2}\left(\dfrac{p_{i,j}^{n+1}-p_{i-1,j}^{n+1}}{y_j-y_{j-1}}\right)}{y_{j+1/2}-y_{j-1/2}}\right]-q_{sc,i,j}^{n+1}B$$

$$=\frac{\Delta x_i\Delta y_j h\left(\phi c_t\right)_{i,j}}{5.615}\left(\frac{p_{i,j}^{n+1}-p_{i,j}^{n}}{\Delta t^{n+1}}\right)$$

Note that $\Delta x_i = x_{i+1/2}-x_{i-1/2}$ and $\Delta y_j = y_{j+1/2}-y_{j-1/2}$

2D Single-Phase Flow Problem

- Define transmissibility in x and y directions and the accumulation term as in the 1-D case considered previously.

$$T_{x,i+1/2,j}\left(p_{i+1,j}^{n+1}-p_{i,j}^{n+1}\right)-T_{x,i-1/2,j}\left(p_{i,j}^{n+1}-p_{i-1,j}^{n+1}\right)$$
$$+T_{y,i,j+1/2}\left(p_{i,j+1}^{n+1}-p_{i,j}^{n+1}\right)-T_{y,i,j-1/2}\left(p_{i,j}^{n+1}-p_{i,j-1}^{n+1}\right)-q_{sc,i,j}^{\;n+1}B=\widetilde{V}_{i,j}\left(p_{i,j}^{n+1}-p_{i,j}^{n}\right)$$

where

$$T_{x,i+1/2,j}=1.127\times10^{-3}\frac{\lambda_{x,i+1/2,j}\Delta y_j h}{\left(x_{i+1}-x_i\right)},\; T_{x,i-1/2,j}=1.127\times10^{-3}\frac{\lambda_{x,i-1/2,j}\Delta y_j h}{\left(x_i-x_{i-1}\right)},$$

$$T_{y,i,j+1/2}=1.127\times10^{-3}\frac{\lambda_{y,i,j+1/2}\Delta x_i h}{\left(y_{j+1}-y_j\right)},\; T_{y,i,j-1/2}=1.127\times10^{-3}\frac{\lambda_{y,i,j-1/2}\Delta x_i h}{\left(y_j-y_{j-1}\right)}$$

$$and\quad \widetilde{V}_{i,j}=\frac{(\phi c_t)_{i,j}\Delta x_i\Delta y_j h}{5.615\Delta t^{n+1}},\quad \lambda_{x,i\pm1/2,j}=\frac{k_{x,i\pm1/2,j}}{\mu_{i\pm1/2,j}},\quad \lambda_{y,i,j\pm1/2}=\frac{k_{y,i,j\pm1/2}}{\mu_{i,j\pm1/2}}$$

2D Single-Phase Flow Problem

- We rearrange the difference equation as the unknowns at the left-hand side and the known ones at the right-hand side. We will order the unknown pressures first in the x-direction (from left to right) and then in the y-direction (from bottom to top):

$$-T_{x,i-1/2,j}\,p_{i-1,j}^{n+1}+\left(T_{x,i-1/2,j}+T_{x,i+1/2,j}+T_{y,i,j-1/2}+T_{y,i,j+1/2}+\tilde{V}_{i,j}\right)p_{i,j}^{n+1}-T_{x,i+1/2,j}\,p_{i+1,j}^{n+1}$$

$$-T_{y,i,j-1/2}\,p_{i,j-1}^{n+1}-T_{y,i,j+1/2}\,p_{i,j+1}^{n+1}=\tilde{V}_{i,j}\,p_{i,j}^{n}-q_{sc,i,j}^{n+1}B$$

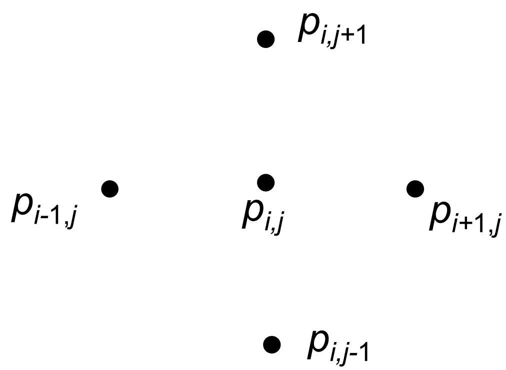
This is the implicit difference equation that applies for both block-centered and point-centered grid. However, the difference equation Needs to be modified at the boundaries depending on the type of boundary conditions.

2D Single-Phase Flow Problem

- Note that the difference equations involve 5 unknown in 2D flow as compared to 3 unknowns in 1D flow we saw previously.

$$-T_{x,i-1/2,j}\,p_{i-1,j}^{n+1}+\left(T_{x,i-1/2,j}+T_{x,i+1/2,j}+T_{y,i,j-1/2}+T_{y,i,j+1/2}+\widetilde{V}_{i,j}\right)p_{i,j}^{n+1}-T_{x,i+1/2,j}\,p_{i+1,j}^{n+1}$$

$$-T_{y,i,j-1/2}\,p_{i,j-1}^{n+1}-T_{y,i,j+1/2}\,p_{i,j+1}^{n+1}=\widetilde{V}_{i,j}\,p_{i,j}^{n}-q_{sc,i,j}^{n+1}B$$



THANK YOU