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Numerical Methods and Reservoir Simulation

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L10: Methods of Solving Systems of Linear Equations (Part 3)

Outline

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Jacobi Method (Iterative Method)

To solve the following system of equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

this method makes two assumptions:

(1) the system has a unique solution, and

(2) the coefficient matrix A has no zeros on its main diagonal.
If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for x_1, the second equation for x_2, and so on, as follows.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})$$

Then make an *initial approximation* of the solution,

$(x_1, x_2, x_3, \ldots, x_n)$, **Initial approximation**

and substitute these values of x_i into the right-hand side of the rewritten equations to obtain the *first approximation*. After this procedure has been completed, one **iteration** has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often **converges** to the actual solution.

Example

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution

To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Because you do not know the actual solution, choose

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0$$ **Initial approximation**

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in Table.

TABLE

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Note that after only five iterations of the Gauss-Seidel method it was achieved the same accuracy as was obtained with seven iterations of the Jacobi method in the Example.

If A is strictly diagonally dominant, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has a unique solution to which the Jacobi method will converge for any initial approximation.

Gauss-Jordan Elimination Method (Direct Method)

Example

Solve the following system by using the Gauss-Jordan elimination method:

$$x + y + z = 5$$
$$2x + 3y + 5z = 8$$
$$4x + 5z = 2$$

Solution

The augmented matrix of the system is the following:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in **reduced row echelon form** (RREF):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Reduced Row Echelon Form (RREF)

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Summary

- The Jacobi method is an iterative method to solve a system of linear equations. A comparison between the Gauss-Seidel method and Jacobi method shows that the solution can be reached faster by the Gauss-Seidel method.

- The Gauss-Jordan Elimination method is a direct method to solve a system of linear equations. The purpose of the method is to obtain a matrix of reduced row echelon form (RREF) to find the solution.

Homework

Use the Jacobi and Gauss-Jordan Elimination methods to solve the following system of equations:

$$1.9\,P1 - 0.45\,P2 = 5125$$

$$-0.45\,P1 + 1.9\,P2 - 0.45\,P3 = 4000$$

$$-0.45\,P2 + 1.9\,P3 = 5800$$

THANK YOU