Al-Ayen University
College of Petroleum Engineering

Numerical Methods and reservoir Simulation

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L6: Principles of Finite Difference Approximation (*Explicit Approximation*)

Outlines

- *1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir*
  - ***Explicit Method***
  - ***Stability of Explicit Method*****Continuity Equation**
  - ***Example - Explicit Method*** **Compressibility Equation**

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

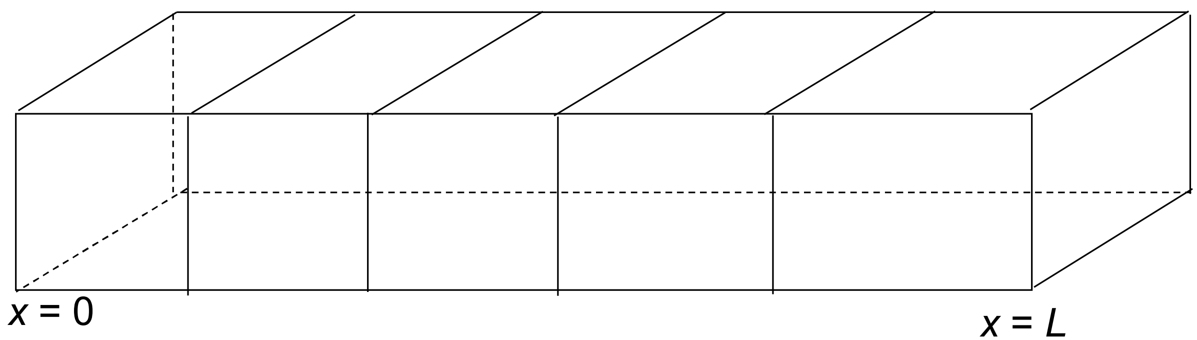
PDE $$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t}, 0 < x < L, t > 0$$

$$\eta = \frac{k}{\phi c_t \mu}$$

IC $$p(x,0) = f(x)\ 0 \le x \le L,$$

BC's $$p(x=0,t) = h(t)\quad t > 0,$$

BC's $$p(x=L,t) = g(t)\quad t > 0,$$



1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Explicit Method

PDE $$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} \quad , \qquad 0 < x < L\,, \quad t > 0$$

Explicit Finite Difference $$\frac{p_{i+1}^{n} - 2p_i^{n} + p_{i-1}^{n}}{(\Delta x)^2} = \frac{1}{\eta}\left(\frac{p_i^{n+1} - p_i^{n}}{\Delta t}\right)$$

The only unknown is p_i^{n+1} and hence the equation can be rearranged to obtain a simple formula:

$$p_i^{n+1} = \alpha\, p_{i-1}^{n} + (1 - 2\alpha)\, p_i^{n} + \alpha\, p_{i+1}^{n}$$

$$\alpha = \frac{\eta \Delta t}{(\Delta x)^2}$$

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Stability of Explicit Method

$$p_i^{n+1} = \alpha\, p_{i-1}^n + (1-2\alpha) p_i^n + \alpha\, p_{i+1}^n$$

- A finite difference scheme will be said stable if any error introduced at a grid point at a given time level does not grow exponentially at later stages of the computations.
- The explicit method is unstable if α does not meet the following requirement:

$$0 < \alpha \le \frac{1}{2} \Rightarrow \alpha = \frac{\eta \Delta t}{\Delta x^2} \le \frac{1}{2} \Rightarrow \Delta t \le \frac{\Delta x^2}{2\eta}$$

$$\eta = \frac{k}{\phi c_t \mu}$$

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Example - Explicit Method

PDE $$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t}, 0 < x < L, t > 0$$

IC $$p(x, t = 0) = 3000\ psia,\ \ 0 \le x \le L$$

BC's $$p(x = 0, t > 0) = 5000\ psia,$$

BC's $$p(x = L, t > 0) = 3000\ psia,$$

Take: L = 1000, and η = 5.0x10^5 ft^2/day, N_x = 10, Δx = 1000/10

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Example - Explicit Method

The analytical solution for this problem is available and is given by

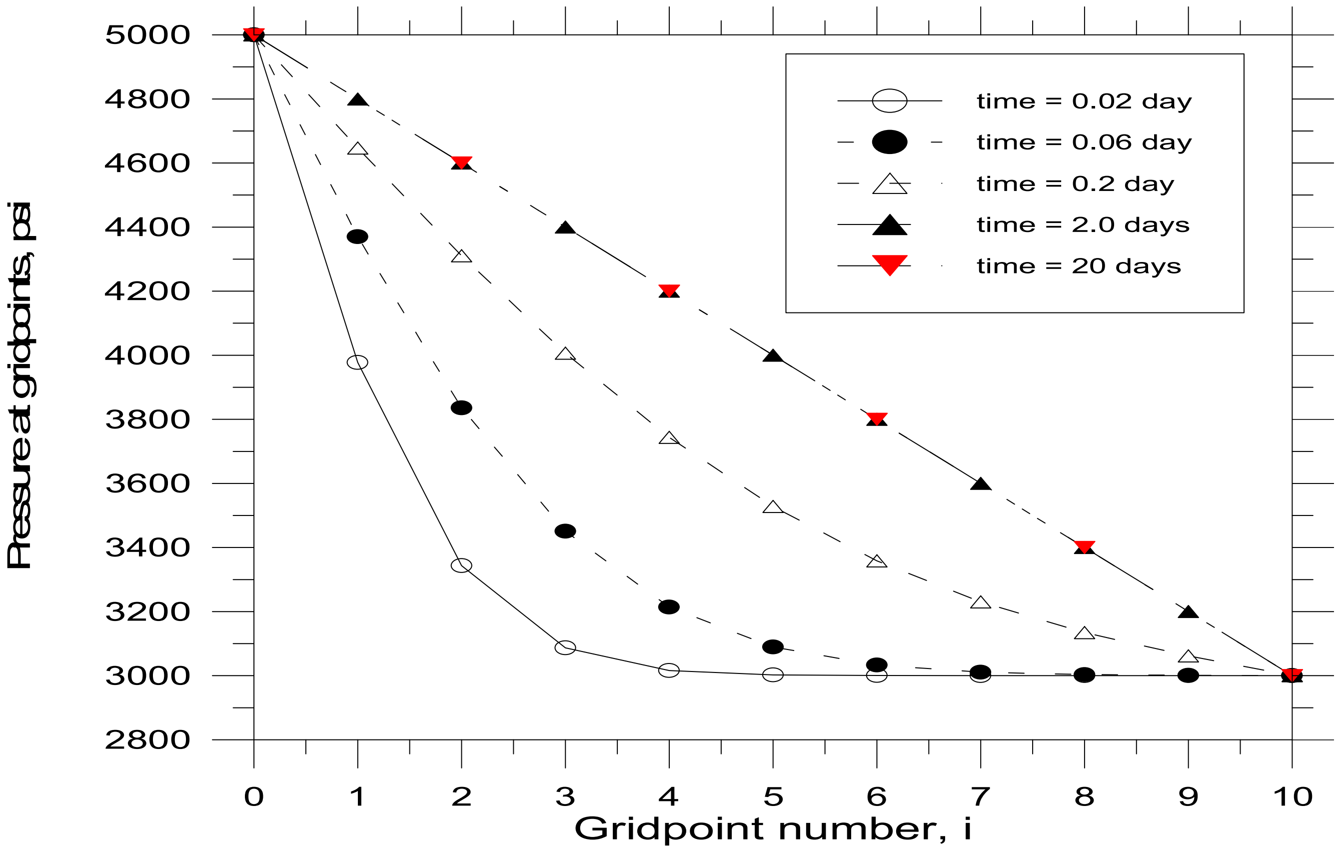
$$p(x,t) = p_L + (p_0 - p_L)\left[\frac{x}{L} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(m\pi\frac{x}{L}\right)\exp\left(-m^2\pi^2\frac{\eta}{L^2}t\right)\right]$$

where p_0 = 3000 psia and p_L = 5000 psia

This serves us to check the accuracy of the numerical solutions computed from explicit method.

Example - Explicit Method

- Results with Δt = 0.002 day. Does this satisfy the stability criterion for explicit scheme?



Example - Explicit Method

- Results with Δt = 0.002 day.

Table 1. Comparison of Analytical and Numerical Results at x = 100 ft

Time, days	Pressure, psia, (Analytical solution)	Pressure, psia, (Explicit numerical solution)	Absolute error, psia	Relative error, percentage
0.02	3959.000	3977.343	18.34	0.46
0.06	4366.183	4369.711	3.53	0.08
0.2	4646.089	4646.667	0.58	0.01
2	4799.980	4799.979	0.001	0.00
20	4800.000	4800.000	0.000	0.00

Example - Explicit Method

- Results with Δt = 0.002 day.

Table 2. Comparison of Analytical and Numerical Results at x = 500 ft

Time, days	Pressure, psia (Analytical solution)	Pressure, psia (Explicit numerical solution)	Absolute error, psi	Relative error, percentage
0.02	3000.814	3002.117	1.30	0.043
0.06	3082.454	3089.404	6.95	0.225
0.2	3525.513	3527.894	2.38	0.068
2	3999.934	3999.933	0.001	0.00
20	4000.000	4000.000	0.000	0.00

Example - Explicit Method

- Results with Δt = 0.02 day. Does this satisfy the stability criterion for explicit scheme?

time= 0.20 day

grid no:		pressure, psia
i=	0	p (0) = 5000.00000000000
i=	1	p (1) = -1201000.00000000
i=	2	p(2) = 1969000.00000000
i=	3	p(3) = -2081000.00000000
i=	4	p(4) = 1699000.00000000
i=	5	p(5) = -1097000.00000000
i=	6	p(6) = 577000.000000000
i=	7	p(7) = -233000.000000000
i=	8	p(8) = 77000.0000000000
i=	9	p(9) = -13000.0000000000
i=	10	p(10)= 3000.00000000000

Unrealistic very large and small/negative numbers,
Sign of instability

- ***Sensitivity to Δt:*** When you make Δt *too big,* the predicted numerical solution of the PDE goes badly wrong - indeed *negative* pressures can occur which is physically impossible. In fact, the solution has become unstable for the larger time steps. This means that this explicit numerical method does have some time step limitations which we must be careful of.

- ***Sensitivity to Δx:*** the effects of grid refinement are that, as the grid blocks get smaller (i.e. $\Delta x \rightarrow 0$ or $NX \rightarrow \infty$). The answer should get more accurate although, to make this happen, you need to reduce the time step as well.

- ***Behaviour as t → ∞ :*** Finally, considering the long-time behavior of the solution of the PDE, you should find that $P(x, t \rightarrow \infty)$ tends to a straight line. As $t \rightarrow \infty$, then if steady-state is reached, then:

$$\left(\frac{\partial P}{\partial t}\right) = 0, \; which\ implies \; => \left(\frac{\partial^2 P}{\partial x^2}\right) = 0$$

THANK YOU