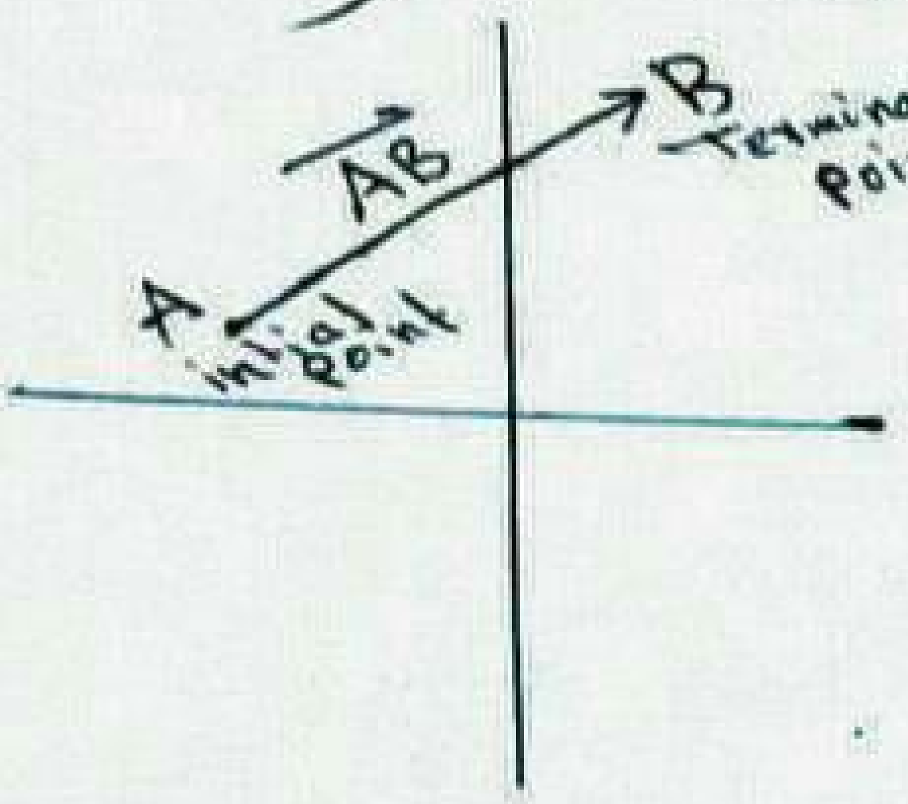
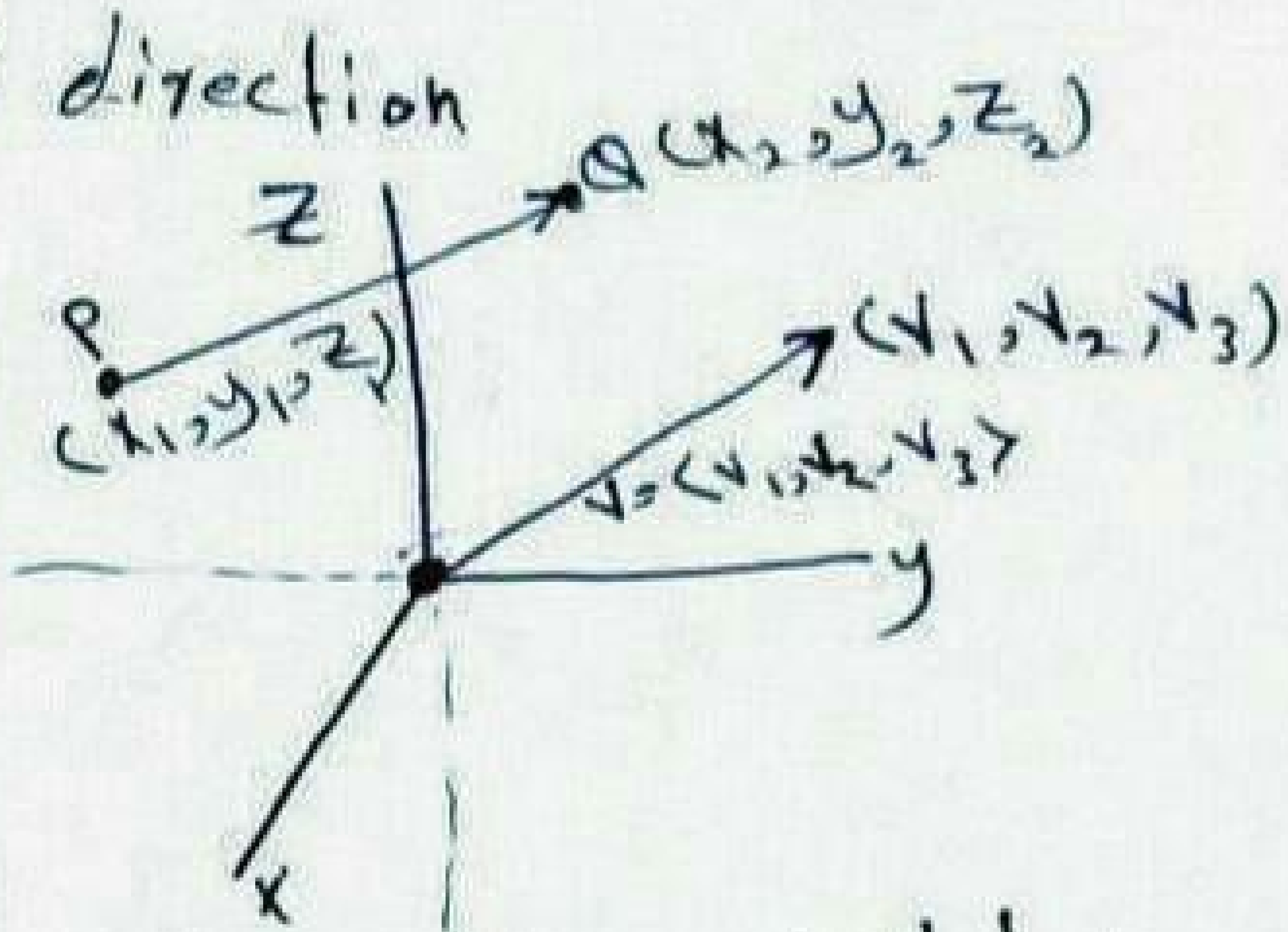
Vectors

Deffintions:- The vector represented by the directed line segment $\overrightarrow{AB}$ has intial point A and terminal point B and its length is denoted by $|\overrightarrow{AB}|$. Two vectors are equal if they have the same length and direction





* If v is a two-dimensional vector in the plane equal to the vector with intial point at the origion and terminal point (v_1, v_2), then the component form of v is

$$v = \langle v_1, v_2 \rangle$$

* If v is a three-dimensional vector equal to the vector with intial point at the origion and terminal point (v_1, v_2, v_3), then the component form of v is

$$v = \langle v_1, v_2, v_3 \rangle$$

* The magnitude or length of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

three-dimension

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

two-dimension

$$|v| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

* The only vector with length 0 is the Zero vector $0 = \langle 0, 0 \rangle$ or

Example 1

Find the a) component form and b) length of the vector with initial point $P(\underset{x_1}{-3}, \underset{y_1}{4})$ and terminal point $Q(\underset{x_2}{-5}, \underset{y_2}{2})$

Solution

$$\underset{\overrightarrow{PQ}}{v} = \langle v_1, v_2 \rangle$$

a)

$$v_1 = x_2 - x_1 = -5 - (-3) = {}^-2$$

$$v_2 = y_2 - y_1 = 2 - 4 = {}^-2$$

The component form of $\overrightarrow{PQ}$ is

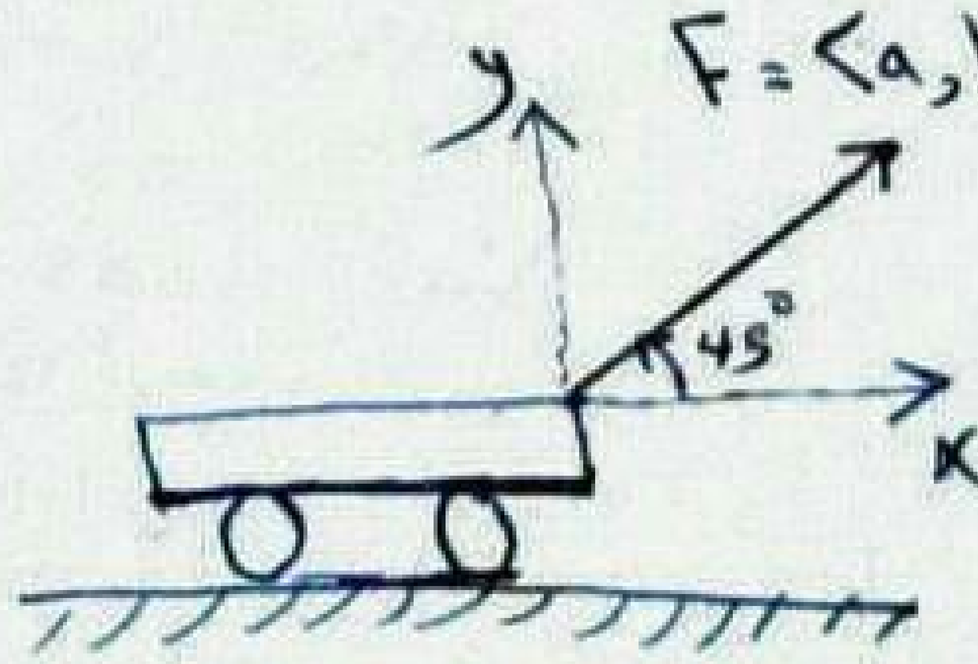
$$\underset{\overrightarrow{PQ}}{v} = \langle -2, {}^-2 \rangle$$

b) The length or magnitude of $v = \overrightarrow{PQ}$ is

$$|v| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

Example 2

A small cart is being pulled along a smooth horizontal floor with a 20 Ib force F making a 45° angle to the floor. What is the effective force moving the cart forward ??



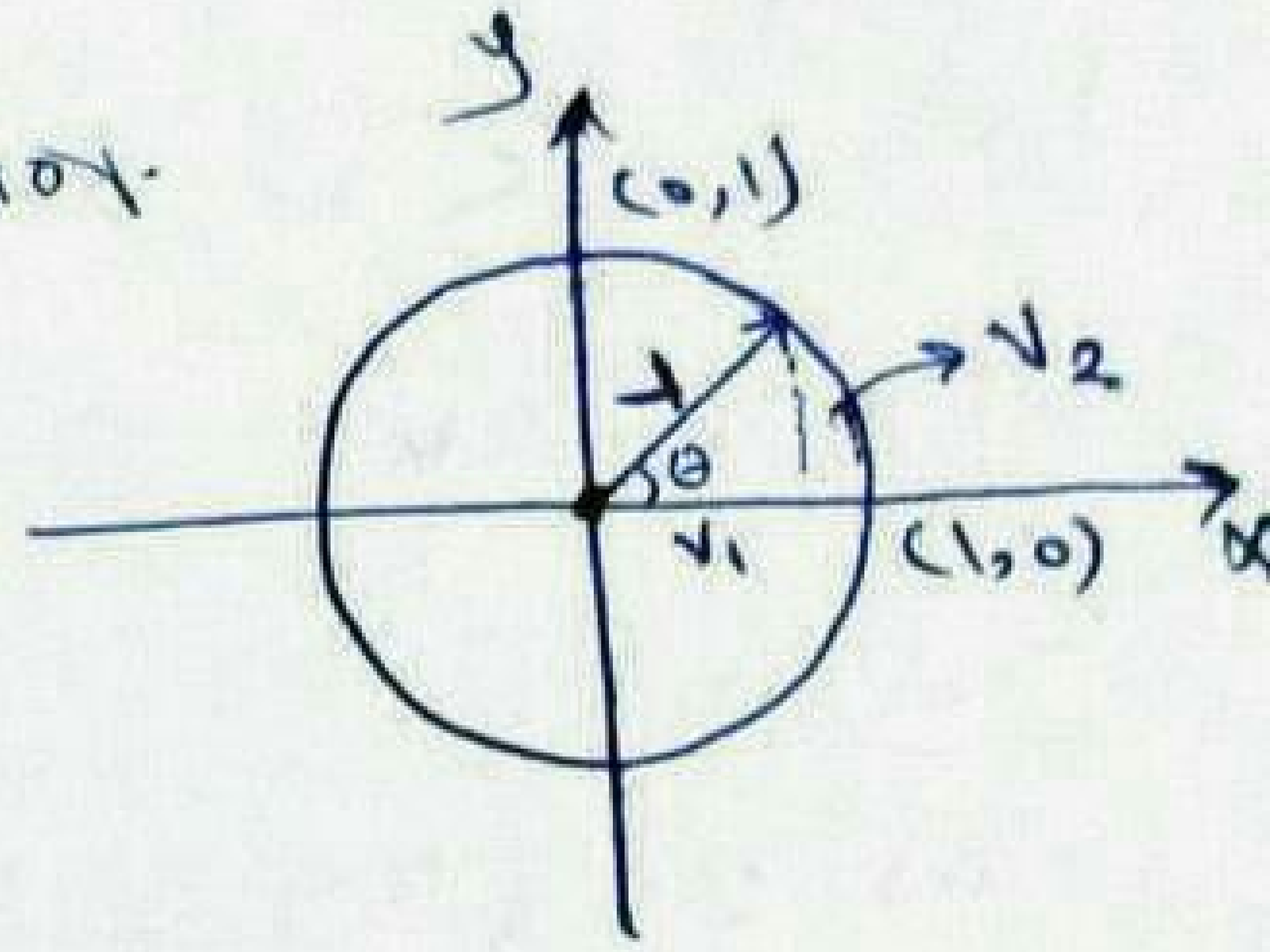
Solution

The effective force is the horizontal component of $F = \langle a, b \rangle$, given by

$$a = |F| \cos 45° = 20 *(1/(\sqrt{2})) \cong 14{,}14 \text{ Ib}$$

Unit Vectors

Any vector v of length 1 is a unit vector.
If $v = \langle v_1, v_2 \rangle$ makes an angle θ with the positive x-axis, then



$v_1 = |v| \cos\theta = \cos\theta \quad , |v| = 1$

$v_2 = |v| \sin\theta = \sin\theta$

A unit vector v in the plane having angle θ with the positive x-axis is represented by $v = \langle \cos\theta, \sin\theta \rangle$

Vector Algebra Operations

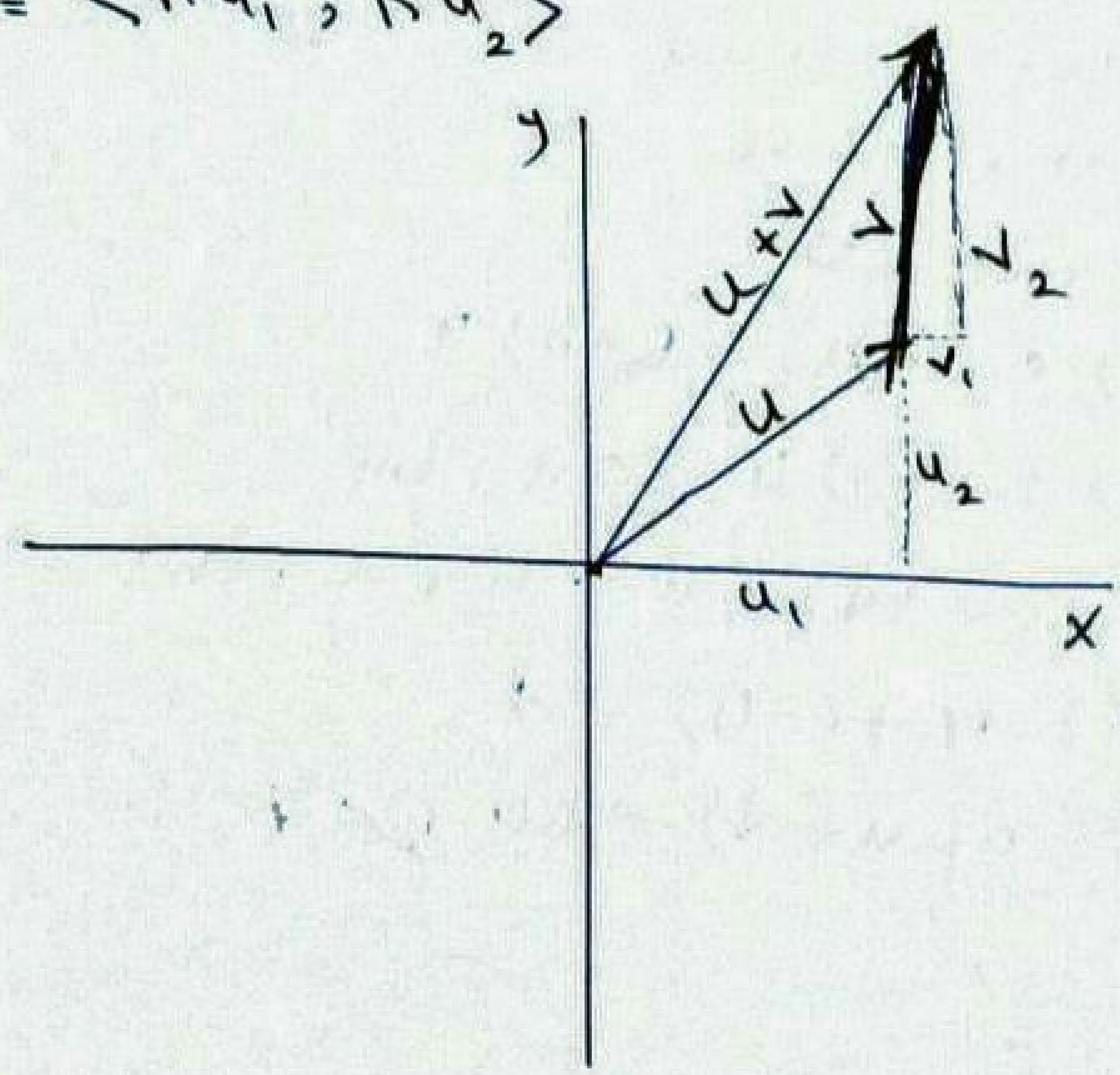
Let $u = \langle u_1, u_2 \rangle$, $v = \langle v_1, v_2 \rangle$ be vectors with

Addition

$$u + v = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$$

multiplication

$$Ru = \langle Ru_1, Ru_2 \rangle$$



Example

Let $u = \langle -1, 3 \rangle$ and $v = \langle 4, 7 \rangle$. Find

ⓐ $2u + 3v$ ⓑ $u - v$ c) $\left| \frac{1}{2} u \right|$

Solution

a) $2u + 3v = 2\langle -1, 3 \rangle + 3\langle 4, 7 \rangle$

$$= \langle 2(-1) + 3(4), 2(3) + 3(7) \rangle = \langle 10, 27 \rangle$$

b) $u - v = \langle \overset{u_1}{-1}, \overset{u_2}{3} \rangle - \langle \overset{v_1}{4}, \overset{v_2}{7} \rangle$

$$= \langle -1 - 4, 3 - 7 \rangle = \langle -5, -4 \rangle$$

c) $\left| \frac{1}{2} u \right| = \left| \left\langle \frac{-1}{2}, \frac{3}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{2}\sqrt{10}$

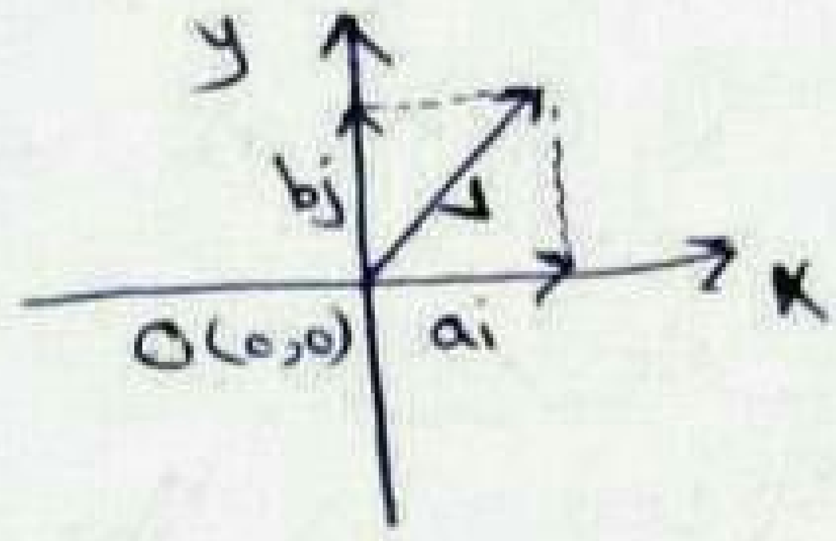
properties of vector operations

Let u, v, w be vectors and a, b be scalars

1) $u + v = v + u$
2) $u + 0 = u$
3) $0u = 0$
4) $a(bu) = (ab)u$
5) $(a+b)u = au + bu$
6) $(u+v) + w = u + (v+w)$
7) $u + (-u) = 0$
8) $a(u+v) = au + av$

Standard Unit Vectors

Any vector $v = \langle a, b \rangle$ in the plane can be written as a linear combination of the two standard unit vectors



$i = \langle 1, 0 \rangle \quad , \quad j = \langle 0, 1 \rangle$

$v = \langle a, b \rangle = ai + bj$

$v = \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle$

$= ai + bj$

* The vector v has slop $\dfrac{b}{a}$

Example

$x_1 \; y_1 \qquad x_2 \; y_2$

Let $P = (-1, 5)$ and $Q(3, 2)$. Write the vector $v = \overrightarrow{PQ}$ as a linear combination of i and j and find its slop.

$(x_2 - x_1) \quad (y_2 - y_1)$

$\langle v_1 , v_2 \rangle$

Solution

The component form of v is $\langle 3 - (-1) , 2 - 5 \rangle = \langle 4, -3 \rangle$

$\therefore \; v = \langle 4, -3 \rangle = 4i + (-3)j = 4i - 3j$

The slop of $v = \dfrac{-3}{4}$

Example

If $v = 3i - 4j$ is a velocity vector, express v as a product of its speed times and a unit vector in the direction of motion.

Solution

speed is the magnitude (length) of v:

$$|v| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = 5$$

The vector

$$\frac{v}{|v|}$$

$$\frac{v}{|v|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$v = 3i - 4j = \underbrace{5}_{\text{length (speed)}}\underbrace{\left(\frac{3}{5}i - \frac{4}{5}j\right)}_{\text{Direction of motion}}$$

$\frac{v}{|v|}$ is a unit vector

$$\left|\frac{v}{|v|}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$$

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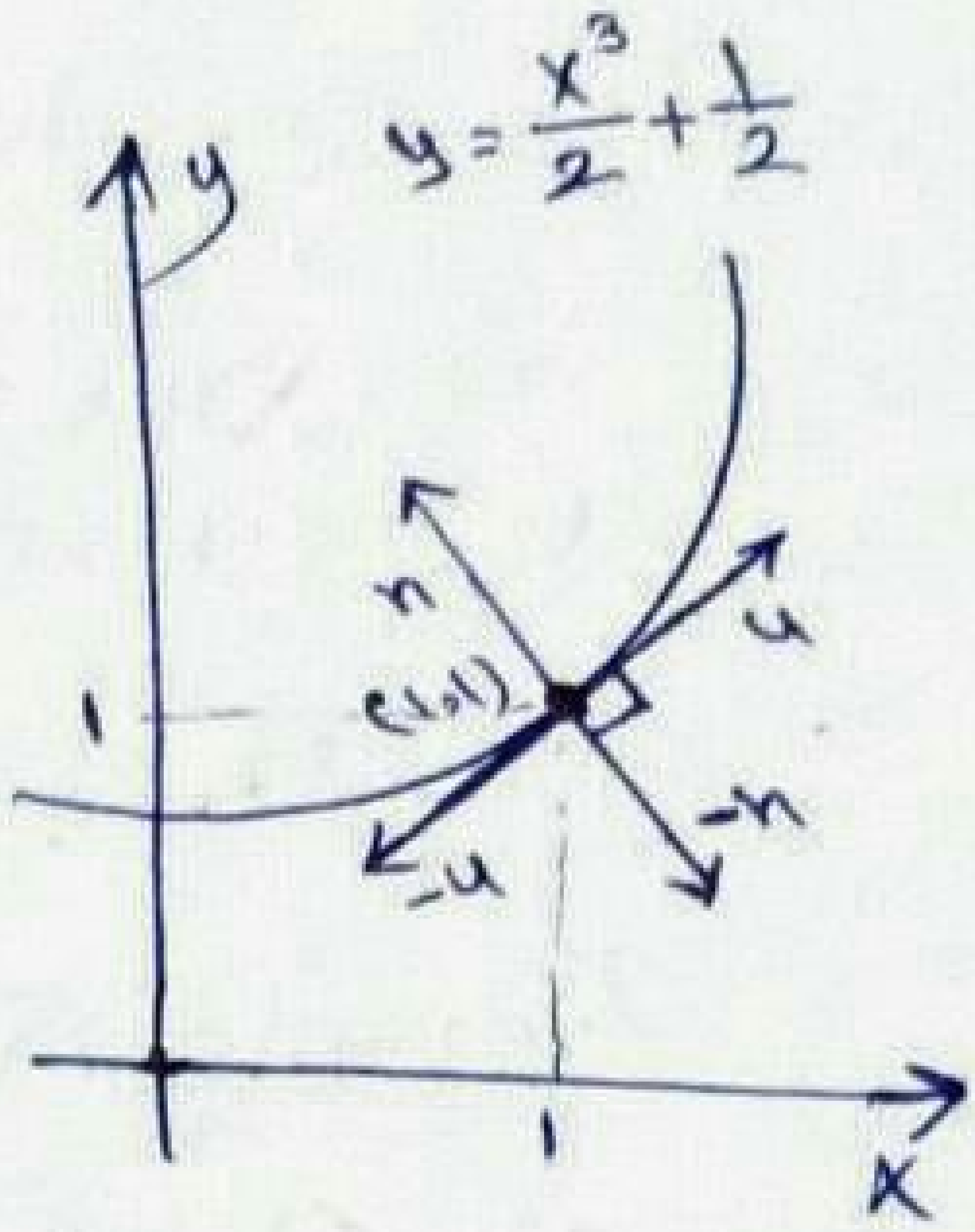
Tangents and Normals

Example



An object is moving along the curve

$$y = \frac{x^3}{2} + \frac{1}{2}$$

Find unit vectors tangent and normal to the curve at the point (1,1)

Solution

- The slope of the line tangents to the curve at (1,1) is

$$\bar{y} = \frac{3x^2}{2}\Big|_{x=1} = \frac{3}{2}$$

- The vector $v = 2i + 3j$ has slope $\frac{3}{2}$
- To Find a multiple of v that is a unit vector

$$|v| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$u = \frac{v}{|v|} = \frac{2}{\sqrt{13}} i + \frac{3}{\sqrt{13}} j$$

$$-u = \frac{-2}{\sqrt{13}} i - \frac{3}{\sqrt{13}} j$$

$$n = \frac{-3}{\sqrt{13}} i + \frac{2}{\sqrt{13}} j$$

$$-n = \frac{3}{\sqrt{13}} i - \frac{2}{\sqrt{13}} j$$

Dot product

The Dot product $(u \cdot v)$ of vectors $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ is the number

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Example: Find the dot product of the following

a) $\langle \underset{u_1}{1}, \underset{u_2}{-2} \rangle \cdot \langle \underset{v_1}{-6}, \underset{v_2}{2} \rangle = (1)(-6) + (-2)(2)$
$= -6 - 4 = -10$

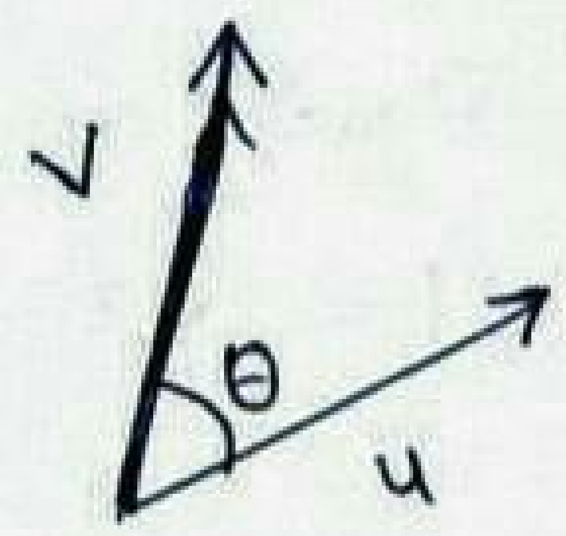
b) $\left(\frac{1}{2}i + 3j\right) \cdot (4i - j) = \left(\frac{1}{2}\right)(4) + (3)(-1)$
$= 2 - 3 = -1$

* Note that the dot product results in a scalar, not a vector

Angle Between Vectors

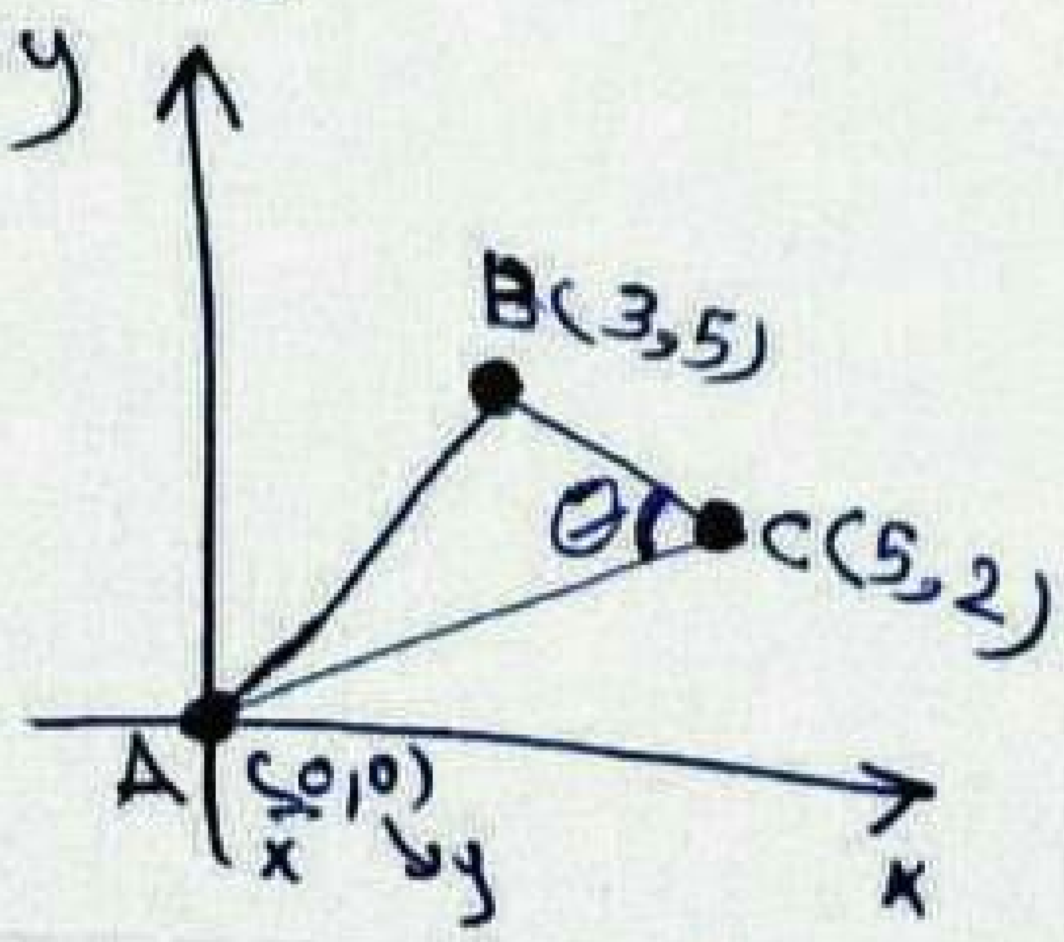
The angle between nonzero vectors u and v is

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u|\,|v|}\right)$$



Example: Find the angle θ in the triangle ABC determined by the vertices

$A = (0,0)$, $B = (3,5)$ and $C = (5,2)$



Solution

The angle θ is the angle between the vertors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

$\overrightarrow{CA} = \langle -5, -2 \rangle$ and $\overrightarrow{CB} = \langle -2, 3 \rangle$

First we calculate the dot product and magnitudes of these two vectors

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

then we find the angle between them

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right)$$

$$\theta = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right) \simeq 78.1^\circ$$

properties of the dot product

If u, v and w are any vectors and α is a scalar, then

1) $u \cdot v = v \cdot u$
2) $(\alpha u) \cdot v = u \cdot (\alpha v) = \alpha (u \cdot v)$
3) $u \cdot (v + w) = u \cdot v + u \cdot w$
4) $u \cdot u = |u|^2$
5) $0 \cdot u = 0$

perpendicular (orthogonal) Vectors

Two nonzero vectors u and v are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$. For such vectors, we automatically have $u.v = 0$ because $\cos(\frac{\pi}{2}) = 0$. If u and v are nonzero vectors with $u.v = |u||v|\cos\theta = 0$, then $\cos\theta = 0$ and

$$\theta = \cos^{-1} 0 = \frac{\pi}{2}$$

Example

a) $u = \langle 3, -2\rangle$ and $v = \langle 4, 6\rangle$ are orthogonal because

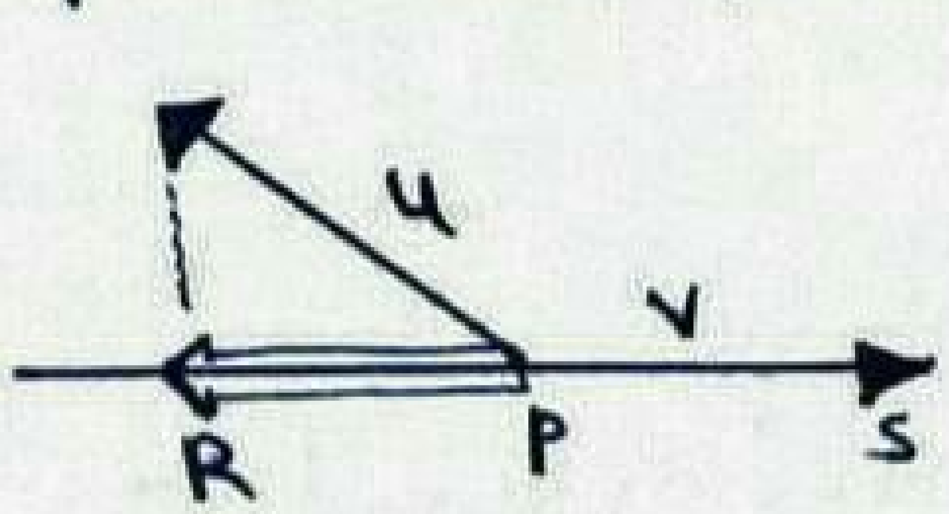
$$u.v = (3)(4) + (-2)(6) = 0$$

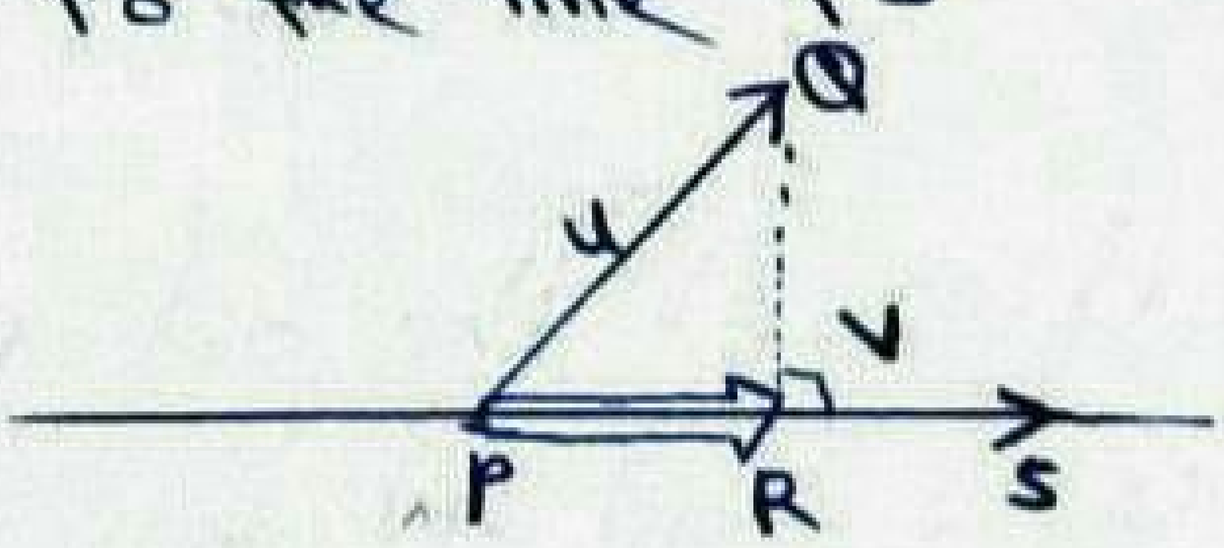
b) $u = i + 2j$ is orthogonal to $v = -10i + 5j$ because

$$u.v = (1)(-10) + (2)(5) = 0$$

Vector projections

The vector projection of $u = \overrightarrow{PQ}$ onto a nonzero vector $v = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular form Q to the line PS





If u represents a force, then $\text{proj}_v u$ the effective force in the direction of v

If the angle θ between u and v is acute, $\text{proj}_v u$ has length $|u|\cos\theta$ and direction $\frac{v}{|v|}$. If θ is obtuse, $\cos < 0$ and $\text{proj}_v u$ has length $-|u|\cos\theta$ and ...