Polar Coordinates

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis (Figure 11.25). The two coordinate systems are then related by the following equations.

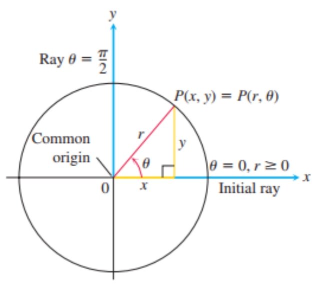

FIGURE 11.25 The usual way to relate polar and Cartesian coordinates.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each $(x, y) \neq (0, 0)$, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two,

EXAMPLE 4 Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 11.26).

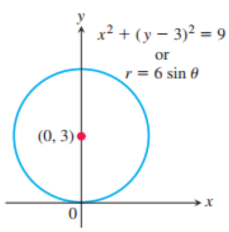

FIGURE 11.26 The circle in Example 5.

Solution We apply the equations relating polar and Cartesian coordinates:

$$\begin{aligned}
x^2 + (y - 3)^2 &= 9 \\
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2,\ y = r\sin\theta \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}$$

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$ Substitution

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x \quad \text{Substitution}$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \quad \text{Completing the square}$$
$$(x-2)^2 + y^2 = 4 \quad \text{Factoring}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$
$$2r\cos\theta - r\sin\theta = 4 \quad \text{Multiplying by } r$$
$$2x - y = 4 \quad \text{Substitution}$$
$$y = 2x - 4 \quad \text{Solve for } y.$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$

Example7 / convert the point $(2,\frac{\pi}{3})$ from polar coordinate to Cartesian coordinate

Example 8 / convert the point (1,-1) from Cartesian coordinate to polar coordinate

Homework

Polar to Cartesian Coordinates

Find the Cartesian coordinates of the following points (given in polar coordinates).

a. $\left(\sqrt{2}, \pi/4\right)$ **b.** $(1, 0)$

c. $(0, \pi/2)$ **d.** $\left(-\sqrt{2}, \pi/4\right)$

Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations.

27. $r\cos\theta = 2$ **28.** $r\sin\theta = -1$

29. $r\sin\theta = 0$ **30.** $r\cos\theta = 0$

31. $r = 4\csc\theta$ **32.** $r = -3\sec\theta$

33. $r\cos\theta + r\sin\theta = 1$ **34.** $r\sin\theta = r\cos\theta$

35. $r^2 = 1$ **36.** $r^2 = 4r\sin\theta$

37. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$ **38.** $r^2\sin 2\theta = 2$

39. $r = \cot\theta\csc\theta$ **40.** $r = 4\tan\theta\sec\theta$

41. $r = \csc\theta\, e^{r\cos\theta}$ **42.** $r\sin\theta = \ln r + \ln\cos\theta$

43. $r^2 + 2r^2\cos\theta\sin\theta = 1$ **44.** $\cos^2\theta = \sin^2\theta$

45. $r^2 = -4r\cos\theta$ **46.** $r^2 = -6r\sin\theta$

47. $r = 8\sin\theta$ **48.** $r = 3\cos\theta$

49. $r = 2\cos\theta + 2\sin\theta$ **50.** $r = 2\cos\theta - \sin\theta$

51. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$ **52.** $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 53–66 with equivalent polar equations.

53. $x = 7$ **54.** $y = 1$ **55.** $x = y$

56. $x - y = 3$ **57.** $x^2 + y^2 = 4$ **58.** $x^2 - y^2 = 1$

59. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **60.** $xy = 2$

61. $y^2 = 4x$ **62.** $x^2 + xy + y^2 = 1$

63. $x^2 + (y - 2)^2 = 4$ **64.** $(x - 5)^2 + y^2 = 25$

65. $(x - 3)^2 + (y + 1)^2 = 4$ **66.** $(x + 2)^2 + (y - 5)^2 = 16$