Areas and Lengths in Polar Coordinates

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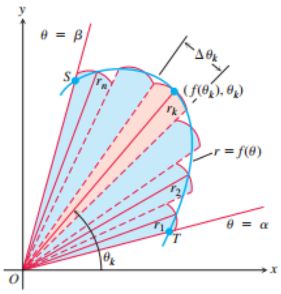


FIGURE To derive a formula for the area of region OTS, we approximate the region with fan-shaped circular sectors.

This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

Area in the Plane

The region OTS in Figure is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are then led to the following formula defining the region's area:

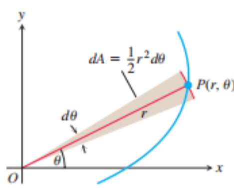


FIGURE The area differential dA for the curve $r = f(\theta)$.

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

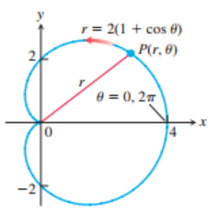


FIGURE The cardioid in Example 1.

EXAMPLE 1 Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$ ■

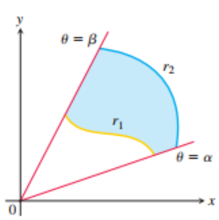


FIGURE The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

To find the area of a region like the one in Figure 11.34, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2 \, d\theta$ from the integral of $(1/2)r_2^2 \, d\theta$. This leads to the following formula.

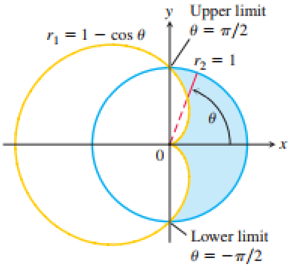


FIGURE The region and limits of integration in Example 2.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 11.35). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\
&= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta && \text{Symmetry} \\
&= \int_{0}^{\pi/2} \left(1 - (1 - 2\cos\theta + \cos^2\theta)\right) d\theta && \text{Square } r_1. \\
&= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.
\end{aligned}$$

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \qquad (2)$$

The parametric length formula, Equation (3) from Section 11.2, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for x and y (Exercise 29).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad (3)$$

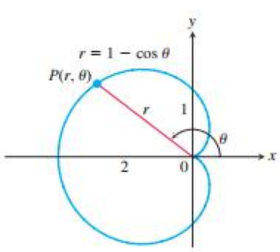


FIGURE Calculating the length of a cardioid (Example 3).

EXAMPLE 3 Find the length of the cardioid $r = 1 - \cos\theta$.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$
$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

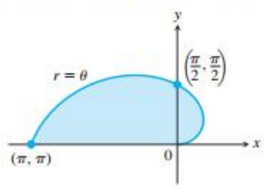
and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$
$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$
$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$
$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

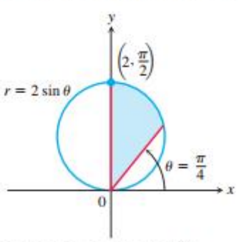
Finding Polar Areas

Find the areas of the regions in Exercises 1–8.

1. Bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$



2. Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$

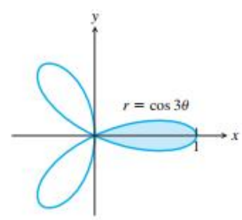


3. Inside the oval limaçon $r = 4 + 2\cos\theta$

4. Inside the cardioid $r = a(1 + \cos\theta),\quad a > 0$

5. Inside one leaf of the four-leaved rose $r = \cos 2\theta$

6. Inside one leaf of the three-leaved rose $r = \cos 3\theta$



7. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$

8. Inside the six-leaved rose $r^2 = 2\sin 3\theta$

Finding Lengths of Polar Curves

Find the lengths of the curves in Exercises 21–28.

21. The spiral $r = \theta^2,\quad 0 \le \theta \le \sqrt{5}$

22. The spiral $r = e^\theta/\sqrt{2},\quad 0 \le \theta \le \pi$

23. The cardioid $r = 1 + \cos\theta$

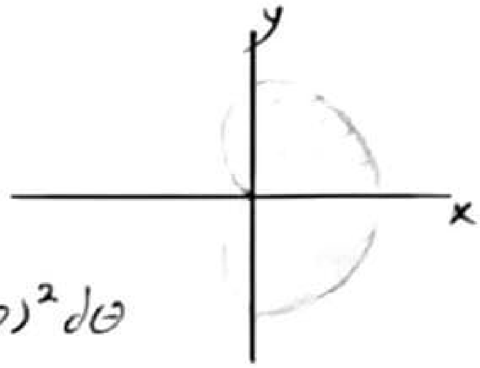
24. The curve $r = a\sin^2(\theta/2),\quad 0 \le \theta \le \pi,\quad a > 0$

25. The parabolic segment $r = 6/(1 + \cos\theta),\quad 0 \le \theta \le \pi/2$

26. The parabolic segment $r = 2/(1 - \cos\theta),\quad \pi/2 \le \theta \le \pi$

Example 1

Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1+\cos\theta)$



Solution

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2} \cdot 4(1+\cos\theta)^2 \, d\theta$$

$$= \int_{0}^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_{0}^{2\pi} (2 + 4\cos\theta + 2\cos^2\theta) \, d\theta$$

$$= \int_{0}^{2\pi} \left(2 + 4\cos\theta + 2\left(\frac{1+\cos 2\theta}{2}\right)\right) d\theta$$

$$= \int_{0}^{2\pi} (2 + 4\cos\theta + 1 + \cos 2\theta) \, d\theta$$

$$= \int_{0}^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta$$

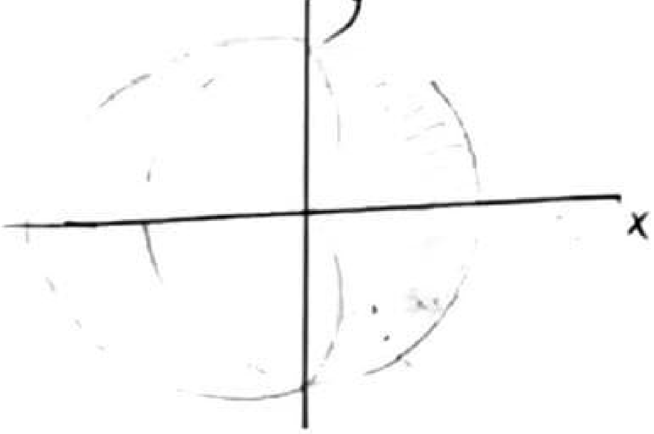
$$= 3\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \Big|_{0}^{2\pi}$$

$$= 3(2\pi - 0) + 4(\sin 2\pi - \sin 0) + \frac{1}{2}(\sin 4\pi - \sin 2(0))$$

$$= 6\pi + 4(0-0) + \frac{1}{2}(0-0) = 6\pi$$

Example 2

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$



Solution

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} ((1)^2 - (1 - \cos\theta)^2)\, d\theta$$

$= \int_{0}^{\frac{\pi}{2}} (1 - 1 + 2\cos\theta$ [redacted] $\cos^2\theta)\, d\theta$

$$= \int_{0}^{\frac{\pi}{2}} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \left(2\cos\theta - \left(\frac{1 + \cos 2\theta}{2}\right)\right) d\theta$$

$$A = \left(2\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)\Big|_{0}^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right) - \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{4}\left(\sin 2\left(\frac{\pi}{2}\right) - \sin 2(0\right.$$

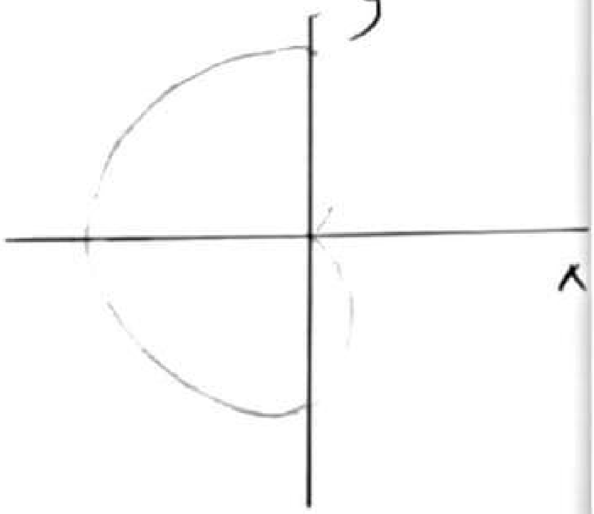
$$= 2(1 - 0) - \frac{\pi}{4} - \frac{1}{4}(0 - 0)$$

$$= 2 - \frac{\pi}{4}$$

Example 3

Find the length of the cardioid

$r = 1 - \cos\theta$



Solution

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$r = 1 - \cos\theta \quad , \quad \frac{dr}{d\theta} = \sin\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= 1 - 2\cos\theta + 1 = 2 - 2\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{(2 - 2\cos\theta)}\, d\theta$$

$$\boxed{\cos 2\alpha = 1 - 2\sin^2\alpha}$$

$$L = \int_{\alpha}^{\beta} \sqrt{2 - 2(1 - 2\sin^2\tfrac{\theta}{2})}\, d\theta = \int_{\alpha}^{\beta} \sqrt{(2 - 2 + 4\sin\tfrac{\theta}{2})}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = \int_{0}^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= -4\cos\frac{\theta}{2}\Big|_{0}^{2\pi} = -4\left[\cos\frac{2\pi}{2} - \cos\frac{0}{2}\right]$$

$$= -4(-1 - 1) = 4 + 4 = 8$$