Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

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L7: Principles of Finite Difference Approximation (*Implicit Approximation*)

Outlines

- *1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir*
  - ***Implicit Method***
    - ***Matrix-Vector Formulation***
  - ***Example - Implicit Method***
- *Summary*

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Implicit Method

PDE $\dfrac{\partial^2 p}{\partial x^2} = \dfrac{1}{\eta}\dfrac{\partial p}{\partial t}$, $0 < x < L\,,\quad t > 0$

Implicit Finite Difference $\dfrac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{(\Delta x)^2} = \dfrac{1}{\eta}\left(\dfrac{p_i^{n+1} - p_i^n}{\Delta t}\right)$

There are three unknowns for each *i*; p_{i-1}^{n+1}, p_i^{n+1}, and p_{i+1}^{n+1}. The equation can be rearranged as:

$$-\alpha\, p_{i-1}^{n+1} + (1+2\alpha)\,p_i^{n+1} - \alpha\, p_{i+1}^{n+1} = p_i^n$$

$$\alpha = \frac{\eta \Delta t}{(\Delta x)^2}$$

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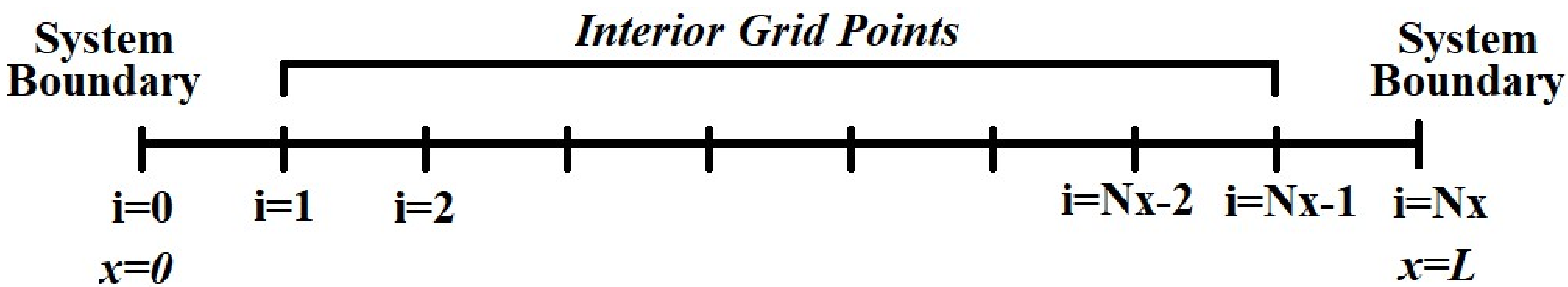
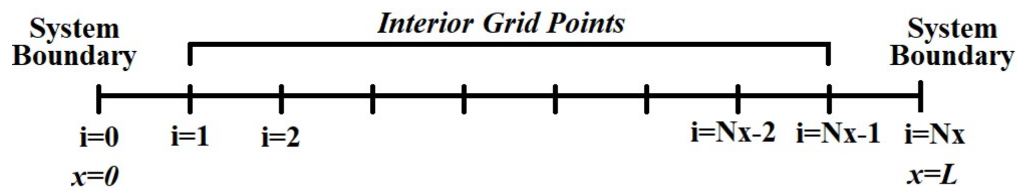


We have N_x-1 unknowns; p_i^{n+1}, for i = 1,2,…,N_x-1; i.e., pressures at the interior grid points.

Matrix-Vector Formulation

$$-\alpha\, p_{i-1}^{n+1} + (1+2\alpha)\, p_i^{n+1} - \alpha\, p_{i+1}^{n+1} = p_i^n$$



For i = 1:

BC

$$-\alpha\, p_0^{n+1} + (1+2\alpha)\, p_1^{n+1} - \alpha\, p_2^{n+1} = p_1^n$$

BC

$$(1+2\alpha)\, p_1^{n+1} - \alpha\, p_2^{n+1} = p_1^n + \alpha p_0^{n+1}$$

For i = 2,3…,N_x-2

$$-\alpha\, p_{i-1}^{n+1} + (1+2\alpha)\, p_i^{n+1} - \alpha\, p_{i+1}^{n+1} = p_i^n$$

For i = N_x-1

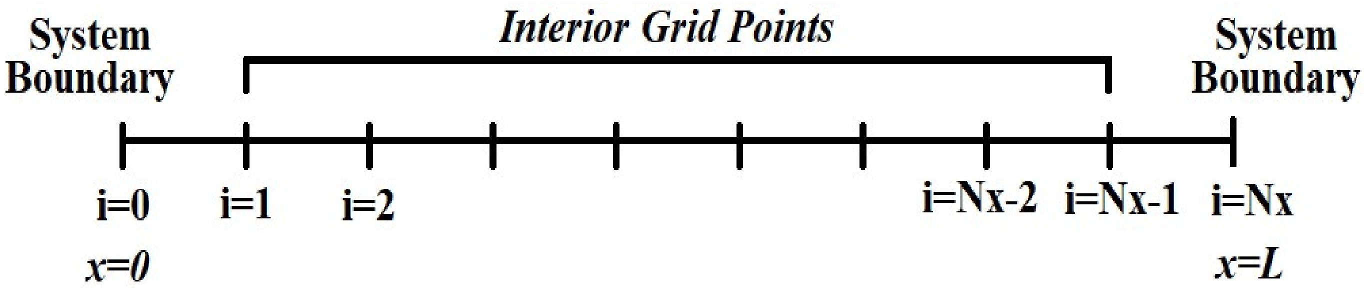
BC

$$-\alpha\, p_{Nx-2}^{n+1} + (1+2\alpha)\, p_{Nx-1}^{n+1} - \alpha\, p_{Nx}^{n+1} = p_{Nx-1}^n$$

BC

$$-\alpha\, p_{Nx-2}^{n+1} + (1+2\alpha)\, p_{Nx-1}^{n+1} = p_{Nx-1}^n + \alpha\, p_{Nx}^{n+1}$$

These equations can be written in a matrix-form: $A\vec{p}^{\,n+1} = \vec{d}^{\,n}$



$$\alpha = \frac{\eta \Delta t}{(\Delta x)^2}$$

$$\begin{bmatrix} (1+2\alpha) & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & (1+2\alpha) & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & (1+2\alpha) & -\alpha & 0 & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & (1+2\alpha) & -\alpha \\ 0 & \cdots & \cdots & 0 & -\alpha & (1+2\alpha) \end{bmatrix} \begin{bmatrix} p_1^{n+1} \\ p_2^{n+1} \\ \vdots \\ p_i^{n+1} \\ \vdots \\ p_{Nx-1}^{n+1} \end{bmatrix} = \begin{bmatrix} p_1^n + \alpha p_0^{n+1} \\ p_2^n \\ \vdots \\ p_i^n \\ \vdots \\ p_{Nx-1}^n + \alpha p_{Nx}^{n+1} \end{bmatrix}$$

$\vec{d}^{\,n}$

A — Coefficient matrix

$\vec{p}^{\,n+1}$ — Vector of unknown pressures at t^{n+1}

$\vec{d}^{\,n}$ — Known vector from t^n.

A is a symmetric, tri-diagonal matrix can be solved by ***direct methods*** or ***iterative methods***

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Example - Implicit Method

PDE $$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t}, 0 < x < L, t > 0$$

IC $$p(x, t = 0) = 3000\ psia,\ \ 0 \le x \le L$$

BC's $$p(x = 0, t > 0) = 5000\ psia,$$

BC's $$p(x = L, t > 0) = 3000\ psia,$$

Take: $L = 1000$, and $\eta = 5.0\text{x}10^5$ ft^2/day, $N_x = 10$, $\Delta x = 1000/10$

1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir

Example - Implicit Method

The analytical solution for this problem is available and is given by

$$p(x,t) = p_L + (p_0 - p_L)\left[\frac{x}{L} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(m\pi\frac{x}{L}\right)\exp\left(-m^2\pi^2\frac{\eta}{L^2}t\right)\right]$$

where p_0 = 3000 psia and p_L = 5000 psia

This serves us to check the accuracy of the numerical solutions computed from implicit method.

Example - Implicit Method

- Results with Δt = 0.002 day.

Table 1. Comparison of Analytical and Numerical Results at x = 100 ft

Time, days	Pressure, psia (Analytical solution)	Pressure, psia (Implicit numerical solution)	Absolute error, psi	Relative error, percentage
0.02	3959.000	3928.613	30.39	0.77
0.06	4366.183	4355.283	10.9	0.25
0.2	4646.089	4644.047	2.042	0.04
2	4799.980	4799.977	0.003	0.00
20	4800.000	4800.000	0.000	0.00

Example - Implicit Method

- Results with Δt = 0.002 day.

Table 2. Comparison of Analytical and Numerical Results at x = 500 ft

Time, days	Pressure, psia (Analytical solution)	Pressure, psia (Implicit numerical solution)	Absolute error, psi	Relative error, percentage
0.02	3000.814	3005.194	4.38	0.146
0.06	3082.454	3092.871	10.42	0.338
0.2	3525.513	3523.418	2.095	0.059
2	3999.934	3999.926	0.008	0.00
20	4000.000	4000.000	0.000	0.00

Example - Implicit Method

- Results with Δt = 0.02 day.

time= 0.20 day

grid no: pressure, psia

i=	0	po(i)=	5000.00000000000
i=	1	po(i)=	4631.57075031308
i=	2	po(i)=	4284.30312188484
i=	3	po(i)=	3975.18171672936
i=	4	po(i)=	3714.36320522494
i=	5	po(i)=	3504.66091729508
i=	6	po(i)=	3342.75413467782
i=	7	po(i)=	3221.20670682680
i=	8	po(i)=	3130.43813178740
i=	9	po(i)=	3060.13982086019
i=	10	po(i)=	3000.00000000000

Abs. Rel. error % = 0.31% (i = 1)

Abs. Rel. error % = 0.59% (i = 5)

It is stable as all computed values are reasonable and realistic.

Summary

- *Explicit* or *Implicit* finite difference approximations can be used to solve PDE of pressure to find the pressure distribution in porous media with respect to time.

- The finite difference approximation is considered *stable if any error introduced at a grid point at a given time level does not grow exponentially at later stages of the computations*.

- The *explicit finite difference* approximation is *conditionally stable*.

- The *implicit finite difference* approximation is *unconditionally stable*.

THANK YOU