Al-Ayen University
College of Petroleum Engineering

# Reservoir Engineering II

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Lecture 7: Unsteady-State Flow of Reservoir Fluids (Part 1)

Ref.: Reservoir Engineering Handbook by Tarek Ahmed

# Outlines

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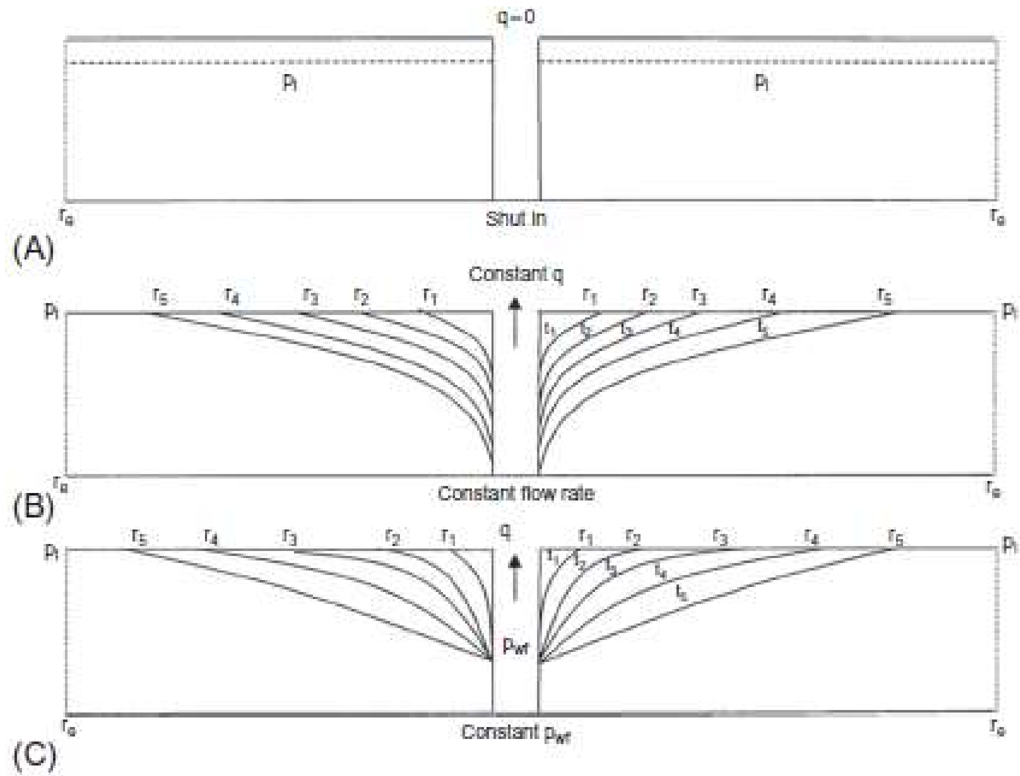
# Unsteady-State Flow

## Introduction

- The transient (unsteady-state) flow is defined as that time period during which the boundary has no effect on the pressure behavior in the reservoir and the reservoir will behave as its infinite in size.

- During this time we say that the reservoir is *infinite acting* because the outer drainage radius $r_e$ can be mathematically *infinite.*




Pressure disturbance as a function of time

# *Unsteady-State Flow*

## Introduction

- Under the *steady-state* flowing condition, the same quantity of fluid enters the flow system as leaves it. In *unsteady-state* flow condition, the flow rate into an element of volume of a porous medium may not be the same as the flow rate out of that element.

- The mathematical formulation of the transient-flow equation is based on combining three independent equations and a specifying set of boundary and initial conditions:

a. **Continuity Equation**

b. **Transport Equation**: Basically, the transport equation is Darcy's equation in its generalized differential form.

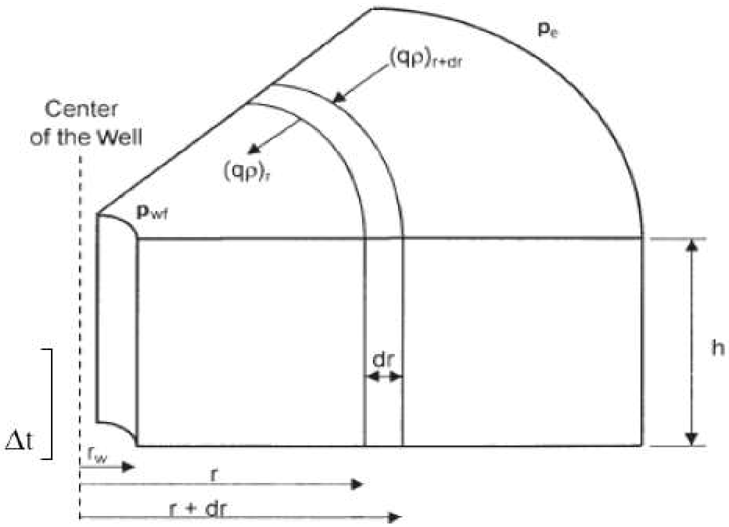
c. **Compressibility Equation (Equation of State)**

d. **Initial and Boundary Conditions**

# *Unsteady-State Flow*

## Radial Flow

### a. Continuity Equation




$$\begin{bmatrix}\text{mass entering}\\ \text{volume element}\\ \text{during interval } \Delta t\end{bmatrix} - \begin{bmatrix}\text{mass leaving}\\ \text{volume element}\\ \text{during interval } \Delta t\end{bmatrix} = \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\\ \text{during interval } \Delta t\end{bmatrix}$$

$$2\pi h\,(r + dr)\,\Delta t\,(\nu\rho)_{r+dr} - 2\pi h r\,\Delta t\,(\nu\rho)_r = (2\pi r h)\,dr\,[(\phi\rho)_{t+\Delta t} - (\phi\rho)_t]$$

Dividing the above equation by $\Delta t(2\pi rh)dr$ and simplifying, gives:

$$\frac{1}{(r)dr}[(r+dr)\,(\nu\rho)_{r+dr} - r\,(\nu\rho)_r] = \frac{1}{\Delta t}\,[(\phi\rho)_{t+\Delta t} - (\phi\rho)_t]$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}[r\,(\nu\rho)] = \frac{\partial}{\partial t}(\phi\rho)}$$

This is the *continuity equation*, and it provides the principle of conservation of mass.

# *Unsteady-State Flow*

## Radial Flow

### b. Transport Equation (Darcy's equation)

$$\nu = (0.006328)\,\frac{k}{\mu}\,\frac{\partial p}{\partial r}$$

where k = permeability, md
ν = velocity, ft/day
p = pressure, psi
r = radius, ft

Substituting the *transport Eq.* into *continuity Eq.* $\boxed{\frac{1}{r}\frac{\partial}{\partial r}[r(\nu\rho)] = \frac{\partial}{\partial t}(\phi\rho)}$, gives:

$$\boxed{\frac{0.006328}{r}\,\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\,\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}(\phi\rho)}$$

The general partial differential equation used to describe the laminar flow of any fluid flowing in a radial direction in porous media.

# *Unsteady-State Flow*

## Radial Flow

### c. Compressibility Equation (Slightly Compressible Fluids)

$$\boxed{\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}(\phi\rho)}$$

The general partial differential equation used to describe the laminar flow of any fluid flowing in a radial direction in porous media.

It needs to separate the change of porosity (rock property) from the change of density (fluid property) with time at R.H.S. of the equation.

$$\frac{\partial}{\partial t}(\phi\rho) = \phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}$$

$$= \phi\frac{\partial\rho}{\partial t} + \rho(\frac{\partial\phi}{\partial p}\frac{\partial p}{\partial t})$$

The *formation compressibility* Cf is the term commonly used to describe the total compressibility of the formation:

$$\boxed{c_f = \frac{1}{\phi}\frac{\partial\phi}{\partial p}} \rightarrow \boxed{\frac{\partial\phi}{\partial p} = c_f\,\phi}$$

$$\frac{\partial}{\partial t}(\phi\rho) = \phi\frac{\partial \rho}{\partial t} + \rho(c_f\,\phi\,\frac{\partial p}{\partial t})$$

$$\boxed{\frac{0.006328}{r}\,\frac{\partial}{\partial r}\left(\frac{k}{\mu}\,(\rho\, r)\frac{\partial p}{\partial r}\right) = \rho\phi c_f\,\frac{\partial p}{\partial t} + \phi\frac{\partial \rho}{\partial t}}$$

To simplify the general equation, assume that the permeability and viscosity are constant over pressure, time, and distance ranges. This leads to:

$$\left[\frac{0.006328\ k}{\mu\ r}\right]\frac{\partial}{\partial r}\left(r\rho\frac{\partial p}{\partial r}\right) = \rho\phi\ c_f\,\frac{\partial p}{\partial t} + \phi\frac{\partial \rho}{\partial t}$$

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\rho}{r}\,\frac{\partial p}{\partial r} + \rho\,\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r}\,\frac{\partial \rho}{\partial r}\right] = \rho\phi\ c_f\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial \rho}{\partial t}\right)$$

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\rho}{r}\,\frac{\partial p}{\partial r} + \rho\,\frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial p}{\partial r}\right)^2\frac{\partial \rho}{\partial p}\right] = \rho\phi\ c_f\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial \rho}{\partial p}\right)\left(\frac{\partial p}{\partial t}\right)$$

Dividing the above expression by the fluid density $\rho$ gives

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{1}{r}\,\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial p}{\partial r}\right)^2\left(\frac{1}{\rho}\,\frac{\partial \rho}{\partial p}\right)\right] = \phi\ c_f\left(\frac{\partial p}{\partial t}\right) + \phi\,\frac{\partial p}{\partial t}\left(\frac{1}{\rho}\,\frac{\partial \rho}{\partial p}\right)$$

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial r^2}+\left(\frac{\partial p}{\partial r}\right)^2\left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)\right]=\phi\, c_f\left(\frac{\partial p}{\partial t}\right)+\phi\frac{\partial p}{\partial t}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)$$

Recalling that the compressibility of any fluid is related to its density by:

$$c=\frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+c\left[\frac{\partial p}{\partial r}\right]^2\right]=\phi\, c_f\left(\frac{\partial p}{\partial t}\right)+\phi\, c\left(\frac{\partial p}{\partial t}\right)$$

The term $c\left(\frac{\partial p}{\partial r}\right)^2$ is considered very small and may be ignored:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right]=\phi(c_f+c)\frac{\partial p}{\partial t}$$

Define total compressibility, $c_t$, as: $c_t = c + c_f$

$$\boxed{\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\phi\mu c_t}{0.006328\,k}\frac{\partial p}{\partial t}}$$

This is the ***diffusivity equation***. It is one of the most important equations in petroleum engineering.

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{0.006328\,k}\frac{\partial p}{\partial t}$$

This ***diffusivity equation*** with the time t is expressed in days.

In analysis of well testing data where the time t is commonly recorded in hours. The equation can be rewritten as:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{0.000264\,k}\frac{\partial p}{\partial t}$$

where $k$ = permeability, md
$r$ = radial position, ft
$p$ = pressure, psia
$t$ = time, hrs
$\phi$ = porosity, fraction
$\mu$ = viscosity, cp
$c_t$ = total compressibility, $\text{psi}^{-1}$
$c_t = c_o S_o + c_w S_w + c_g S_g + c_f$

The ***assumptions*** and ***limitations*** for this equation:

1. *Homogeneous and isotropic porous medium*
2. *Uniform thickness*
3. *Single phase flow*
4. *Laminar flow*
5. *Rock and fluid properties independent of pressure*

Note that the introduction of **C**t into the Eq. does not make the Eq. applicable to multiphase flow; the use of **C**t, simply accounts for the compressibility of any *immobile* fluids that may be in the reservoir with the fluid that is flowing.

# *Unsteady-State Flow*

## Radial Flow

### c. Compressibility Equation (Compressible Fluids)

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}(\phi\rho)$$

The general partial differential equation used to describe the laminar flow of any fluid flowing in a radial direction in porous media.

$$\text{RHS} = \frac{d(\rho\phi)}{dt} = \rho\frac{d\phi}{dt} + \phi\frac{d\rho}{dt}$$

$$= \rho\frac{d\phi}{dP}\frac{dP}{dt} + \phi\frac{d\rho}{dP}\frac{dP}{dt}$$

$$= \rho\phi\left[\left(\frac{1}{\phi}\frac{d\phi}{dP}\right) + \left(\frac{1}{\rho}\frac{d\rho}{dP}\right)\right]\frac{dP}{dt}$$

$$= \rho\phi(c_\phi + c_f)\frac{dP}{dt},$$

where

$c_f$ is the compressibility of the fluid,

$c_\phi$ is the compressibility of the rock

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\frac{\partial p}{\partial r}\right) = \rho\phi(c_\phi + c_f)\frac{\partial p}{\partial t}$$

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\frac{\partial p}{\partial r}\right)=\rho\phi(c_\phi+c_f)\frac{\partial p}{\partial t}$$

Real density equation: $\rho=\dfrac{pM}{zRT}$

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}\left(\frac{pM}{zRT}r\right)\frac{\partial p}{\partial r}\right)=\left(\frac{pM}{zRT}\right)\phi(c_\phi+c_f)\frac{\partial p}{\partial t}$$

The model of gas flow can be derived by making the following assumptions:

(a) The temperature is constant.
(b) The permeability is independent of the pore pressure.

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p}{\mu z}\frac{\partial p}{\partial r}\right)=\frac{\phi\mu c_t}{0.000264\,k}\frac{p}{\mu z}\frac{\partial p}{\partial t}}$$

where $t$ = time, hr

$k$ = permeability, md

$c_t$ = total isothermal compressibility, $psi^{-1}$

$\phi$ = porosity

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p}{\mu z}\frac{\partial p}{\partial r}\right)=\frac{\phi\mu c_t}{0.000264\,k}\frac{p}{\mu z}\frac{\partial p}{\partial t}}$$

Al-Hussainy, Ramey, and Crawford (1966) linearize the above basic flow equation by introducing the real gas potential m(p) :

$$m(p)=\int_0^p\frac{2p}{\mu z}dp$$

Differentiating the above relation with respect to p and r gives:

$$\frac{\partial m(p)}{\partial p}=\frac{2p}{\mu z}$$

$$\frac{\partial m(p)}{\partial r}=\frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{\partial p}{\partial r}=\frac{\mu z}{2p}\frac{\partial m(p)}{\partial r}$$

$$\frac{\partial m(p)}{\partial t}=\frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial t} \quad \Rightarrow \quad \frac{\partial p}{\partial t}=\frac{\mu z}{2p}\frac{\partial m(p)}{\partial t}$$

Combining the previous equations yields:

$$\boxed{\frac{\partial^2 m(p)}{\partial r^2}+\frac{1}{r}\frac{\partial m(p)}{\partial r}=\frac{\phi\mu c_t}{0.000264\,k}\frac{\partial m(p)}{\partial t}} \quad \leftrightarrow \quad \boxed{\text{The radial diffusivity equation for compressible fluids}}$$

# *THANK YOU*