Al-Ayen University
Petroleum Engineering College

Mechanics

Dr. Mohaimen Al-Thamir

Title: centers of gravity; curved shapes
&
moment of inertia

Lec. Number: 5

Determination of centroid by integration

The centroid of an area bounded by analytical curves (i.e., curves defined by algebraic equations) is usually determined by evaluating the integrals. Denoting by $\bar{x}_{el}$ and $\bar{y}_{el}$ the coordinates of the centroid of the element dA, we write

$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA$$

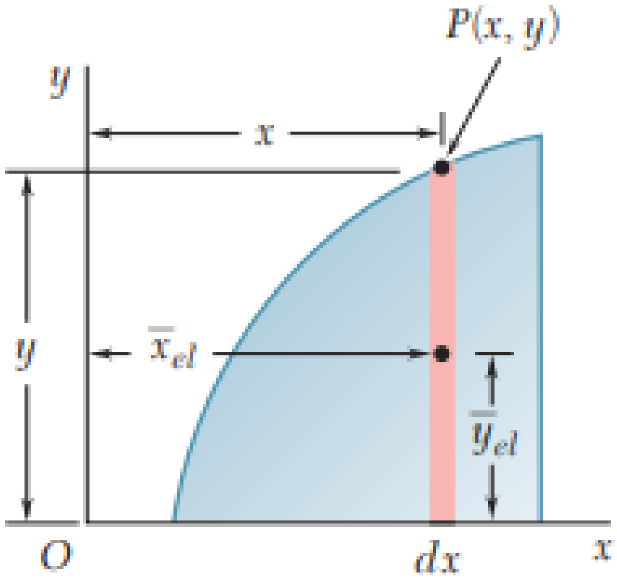
If the area A is not already known, it can also be computed from these elements

When a line is defined by an algebraic equation, its centroid can be determined by evaluating the integrals

$$\bar{x}L = \int x\, dL \qquad \bar{y}L = \int y\, dL$$

Centroids and areas of differential elements

Vertical element

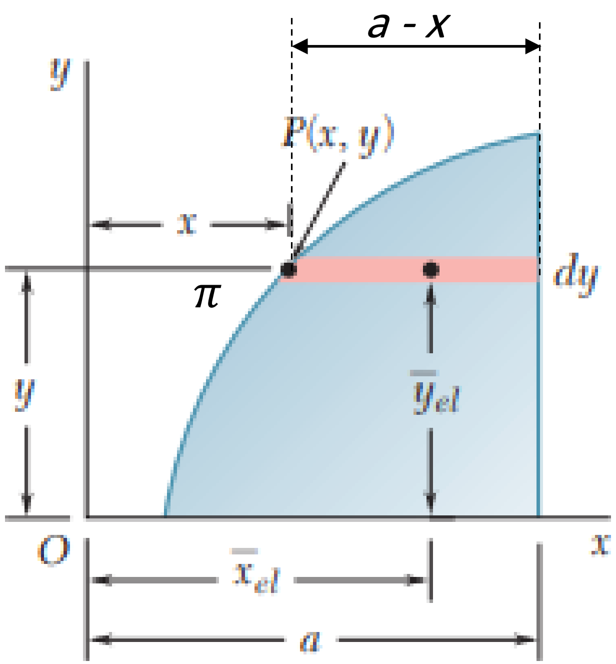


$$\bar{x}_{el} = x$$

$$\bar{y}_{el} = y/2$$

$$dA = y\,dx$$

Horizontal element

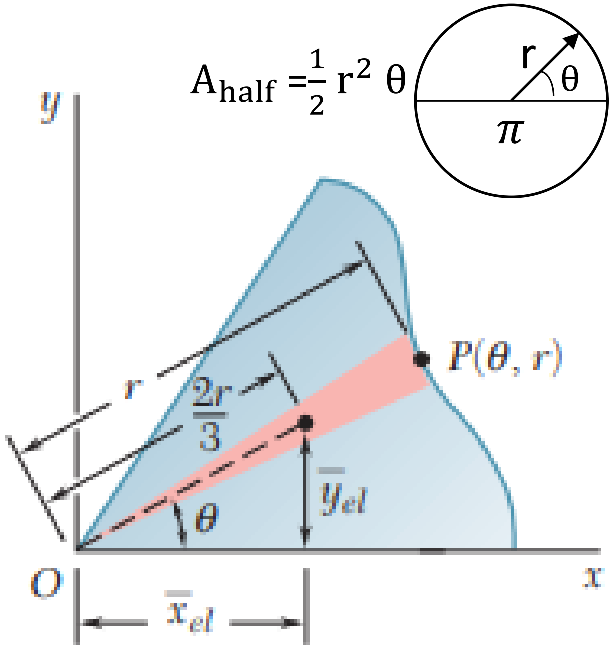


$$\bar{x}_{el} = \frac{a + x}{2}$$

$$\bar{y}_{el} = y$$

$$dA = (a - x)\,dy$$

Polar element

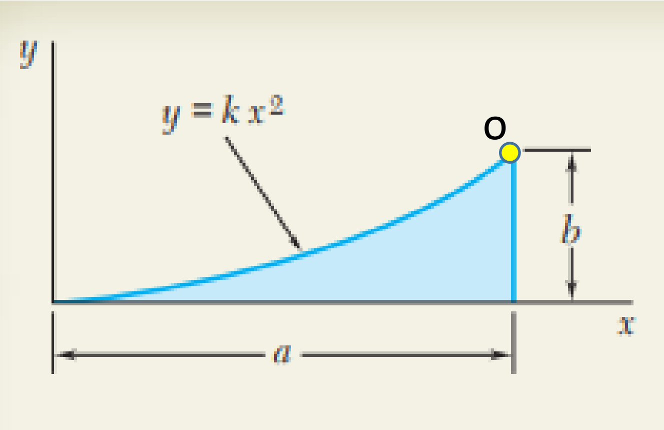


$$\bar{x}_{el} = \frac{2r}{3}\cos\theta$$

$$\bar{y}_{el} = \frac{2r}{3}\sin\theta$$

$$dA = \frac{1}{2}\,r^2\,d\theta$$

Example

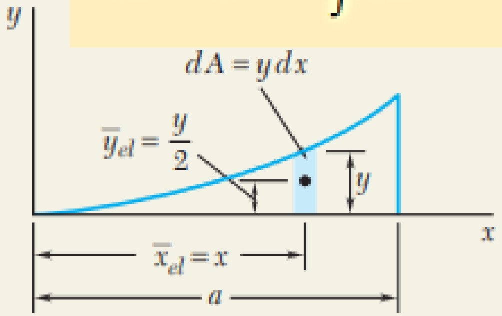


Determine by direct integration the location of the centroid of a parabolic spandrel.

Solution 1

$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA$$



Determination of the Constant k. The value of k is determined by substituting $x = a$ and $y = b$ into the given equation. We have $b = ka^2$ or $k = b/a^2$. The equation of the curve is thus

$$y = \frac{b}{a^2}x^2 \qquad \text{or} \qquad x = \frac{a}{b^{1/2}}y^{1/2}$$

Vertical Differential Element. We choose the differential element shown and find the total area of the figure.

$$A = \int dA = \int y\, dx = \int_0^a \frac{b}{a^2}x^2\, dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a = \frac{ab}{3}$$

The first moment of the differential element with respect to the y axis is $\bar{x}_{el}\, dA$; hence, the first moment of the entire area with respect to this axis is

$$Q_y = \int \bar{x}_{el}\, dA = \int xy\, dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2b}{4}$$

Since $Q_y = \bar{x}A$, we have

$$\bar{x}A = \int \bar{x}_{el}\, dA \qquad \bar{x}\frac{ab}{3} = \frac{a^2b}{4} \qquad \bar{x} = \tfrac{3}{4}a \blacktriangleleft$$

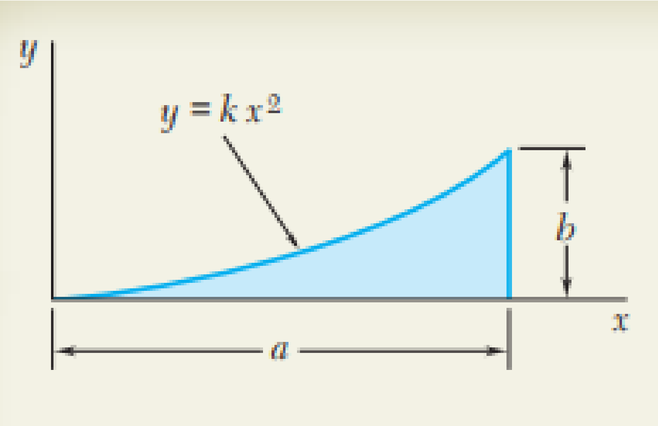
Likewise, the first moment of the differential element with respect to the x axis is $\bar{y}_{el}\, dA$, and the first moment of the entire area is

$$Q_x = \int \bar{y}_{el}\, dA = \int \frac{y}{2}y\, dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Since $Q_x = \bar{y}A$, we have

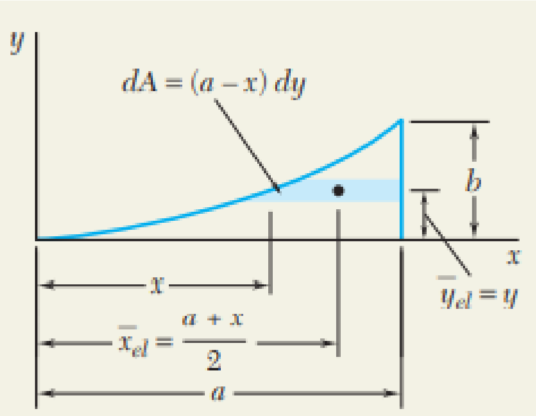
$$\bar{y}A = \int \bar{y}_{el}\, dA \qquad \bar{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \bar{y} = \tfrac{3}{10}b \blacktriangleleft$$

Example



Determine by direct integration the location of the centroid of a parabolic spandrel.

Solution 2



Horizontal Differential Element. The same results can be obtained by considering a horizontal element. The first moments of the area are

$$Q_y = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2-x^2}{2}dy$$

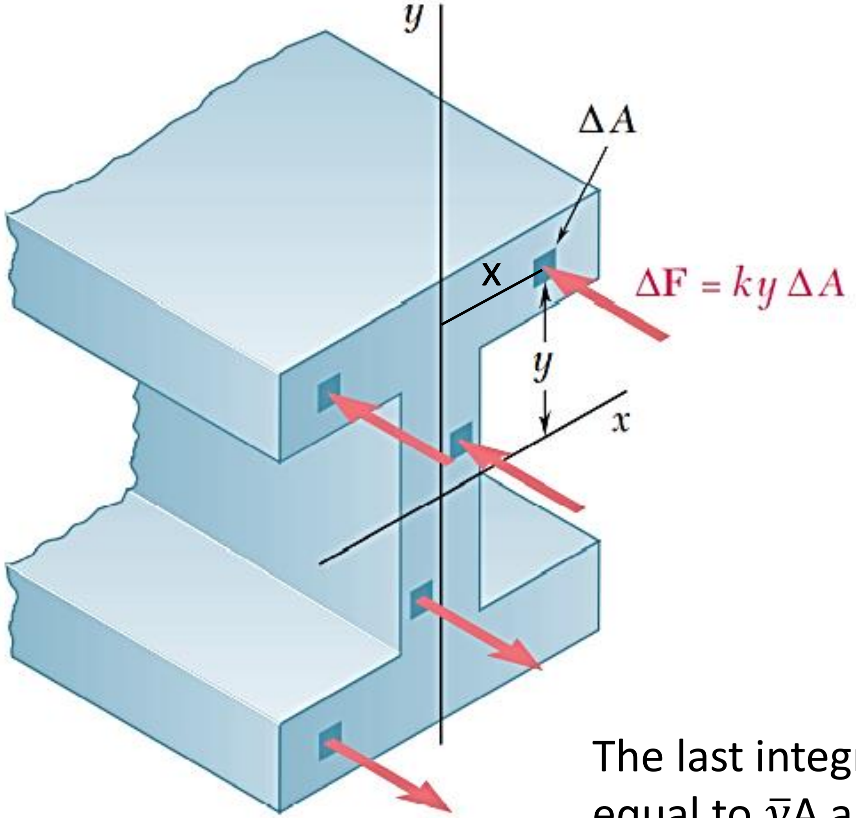
$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\,dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$

To determine $\bar{x}$ and $\bar{y}$, the expressions obtained are again substituted into the equations defining the centroid of the area.

Second moment, or moment of inertia, of an area



Consider, for example, a beam of uniform cross section where the internal forces in any section of the beam are distributed forces whose magnitudes ΔF = k y ΔA vary linearly with the distance y between the element of area ΔA and an axis passing through the centroid of the section. This axis, represented by the x axis in the given figure, is known as the **neutral axis** of the section. The forces on one side of the neutral axis are forces of compression, while those on the other side are forces of tension; on the neutral axis itself the forces are zero.

The magnitude of the resultant R of the elemental forces ΔF which act over the entire section is:

$$R = \int ky\,dA = k\boxed{\int y\,dA}$$

The last integral obtained is recognized as the first moment Q_x of the section about the x axis; it is equal to $\bar{y}$A and is thus equal to zero, since the centroid of the section is located on the x axis. The system of the forces ΔF thus reduces to a couple. The magnitude M of this couple (bending moment) must be equal to the sum of the moments of the elemental forces. Integrating over the entire section, we obtain

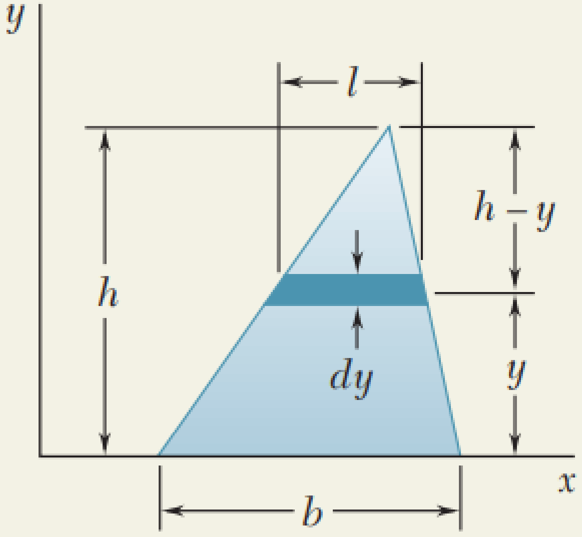
$$M = \int ky^2\,dA = k\boxed{\int y^2\,dA}$$

The last integral is known as the second moment, or moment of inertia, of the beam section with respect to the x axis and is denoted by $\boldsymbol{I_x}$.

Similarly, we can derive the moment of inertia (I_y) of the area A with respect to the y axis. These are summarized by the following set of equations:

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA$$

Example: Determine the moment of inertia of a triangle with respect to its base.



A triangle of base b and height h is drawn; the x axis is chosen to coincide with the base. A differential strip parallel to the x axis is chosen to be dA. Since all portions of the strip are at the same distance from the x axis, we write

$$dI_x = y^2\,dA \qquad dA = l\,dy$$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\,dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$I_x = \frac{bh^3}{12}$$