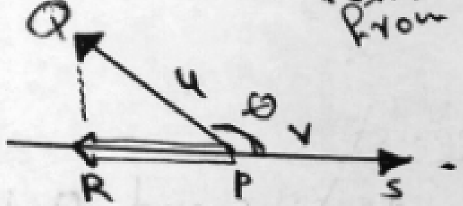
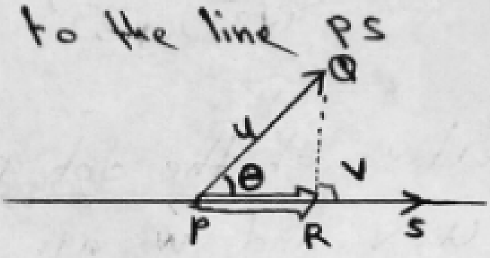
Vector projections

The vector projection of $u = \overrightarrow{PQ}$ onto a nonzero vector $v = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS





If u represents a force, then $\text{proj}_v u$ the effective force in the direction of v

If the angle θ between u and v is acute, $\text{proj}_v u$ has length $|u| \cos\theta$ and direction $\frac{v}{|v|}$. If θ is obtuse, $\cos\theta < 0$ and $\text{proj}_v u$ has length $-|u| \cos\theta$ and direction

Example: Find the vector projection of a force $F = 5i + 2j$ onto $v = i - 3j$ and the scalar component of F in the direction of v.

Solution The vector projection is

$$\text{proj}_v F = \left(\frac{F \cdot v}{|v|^2}\right) v = \frac{(5*1) + (2*-3)}{\left(\sqrt{(1)^2 + (-3)^2}\right)^2} (i - 3j)$$

$$= \frac{5-6}{1+9}(i - 3j) = \frac{-1}{10}(i - 3j)$$

$$= \frac{-1}{10} i + \frac{3}{10} j$$

The scalar component of F in the direction of v is

$$|F| \cos\theta = \frac{F \cdot v}{|v|} = \frac{5-6}{\sqrt{1+9}} = \frac{-1}{\sqrt{10}}$$

Example: show that the vector $v = ai + bj$ is perpendicular to the line $ax + by = c$.

Solution P & Q are any two points on the line

$$ax + by = c \Rightarrow y = \frac{c}{b} - \frac{a}{b}x \rightarrow y_1 = \frac{c}{b} - \frac{a}{b}x_1$$

$$y_2 = \frac{c}{b} - \frac{a}{b}x_2$$

$$\therefore P = (x_1, y_1) = \left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$$

$$Q = (x_2, y_2) = \left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$$

$\frac{c}{b} - \frac{a}{b}x_2 - \left(\frac{c}{b} - \frac{a}{b}x_1\right)$

$- \frac{a}{b}x_2 + \frac{a}{b}x_1$

$= -\frac{a}{b}(x_2 - x_1)$

$$\therefore \overrightarrow{PQ} = (x_2 - x_1)i - \frac{a}{b}(x_2 - x_1)j$$

$$\overrightarrow{PQ} \cdot v = a(x_2 - x_1) + b\left(\frac{-a}{b}\right)(x_2 - x_1) = 0$$

$u \cdot v = |u||v|\cos$
$= 0$

Example Show that the vector $v = ai + bj$ is parallel to the line $bx - ay = c$.

Solution P & Q are any two points on the line

$$bx - ay = c \Rightarrow y = \frac{-c}{a} + \frac{b}{a}x$$

$$P = (x_1, y_1) = \left(x_1, \frac{-c}{a} + \frac{b}{a}x_1\right)$$

$$Q = (x_2, y_2) = \left(x_2, \frac{-c}{a} + \frac{b}{a}x_2\right)$$

$$\vec{PQ} = (x_2 - x_1)\,i + \frac{b}{a}(x_2 - x_1)\,j$$

but the slope of vector v is $\frac{b}{a}$

$\therefore$ the $\vec{PQ}$ and v are parallel.

$$\boxed{\text{slop} = \frac{b}{a}}$$

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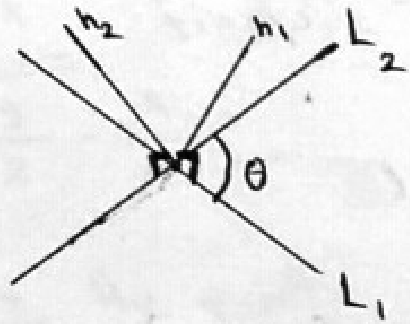
$v = ai + bj$

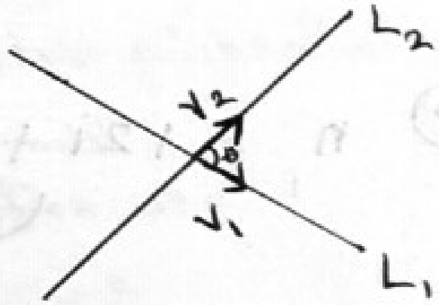
$\text{slop} = \frac{b}{a}$

$\frac{b}{}\,(x_2 - x_1)$

Angles Between Lines in the plane

The acute angle between intersecting line that do not cross at right angles is the same as the angle determined by vectors $\boxed{\text{normal}}$ to the lines or by the vectors parallel to the lines





Ex Find the acute angles between the lines

① $3x+y=5$ & $2x-y=4$

② $12x+5y=1$ & $2x-2y=3$

Sol.

① $3x+y=5 \longrightarrow ax+by=c$

$v=ai+bj$

$\boxed{n_1 = 3i+j}$

$2x-y=4 \longrightarrow ax+by=c$

$v=ai+bj$

$\boxed{n_2 = 2i-j}$

$u \cdot v = |u||v|\cos\theta$

$\cos\theta = \dfrac{u \cdot v}{|u||v|} \Longrightarrow \theta = \cos^{-1}\dfrac{u \cdot v}{|u||v|}$

$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1|\,|n_2|}\right)$$

$$\theta = \cos^{-1} \frac{(3*2)+(1*\bar{1})}{\sqrt{(3)^2+(1)^2} * \sqrt{(2)^2+(-1)^2}}$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{2}} \quad = \frac{\pi}{4}$$

(2) $n_1 = 12i + 5j \longrightarrow \underbrace{12}_{a}x + \underbrace{5}_{b}y = 1$

$$v = ai + bj$$

$n_2 = 2i - 2j \longrightarrow \underbrace{2}_{a}x - \underbrace{2}_{b}y = 3$

$$v = ai + bj$$

$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1|\,|n_2|}\right)$$

$$= \cos^{-1}\left(\frac{(12*2)+(5*\bar{2})}{\sqrt{(12^2)+(5)^2} * \sqrt{2^2+2^2}}\right)$$

$$\theta = \cos \frac{14}{26\sqrt{2}} = 1.18 \text{ rad}$$

Angles Between Diffentible curves

The angles between two differentiable curves at apoint of intersection are the angles between the curves tangent lines at these poi

Ex Find the angle between the curves

$y = \frac{3}{2} - x^2$, $y = x^2$ (two points of intersecti

Soln the points of intersection are

$y = \frac{3}{2} - x^2$, $y = x^2$

$\frac{3}{2} - x^2 = x^2$

$x^2 + x^2 = \frac{3}{2} \rightarrow 2x^2 = \frac{3}{2} \Rightarrow x^2 = \frac{3}{4} \Rightarrow x = \mp \frac{\sqrt{3}}{2}$

$y = \frac{3}{2} - x^2 \Rightarrow y = \frac{3}{2} - (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$

$\hookrightarrow y = \frac{3}{2} - (\frac{-\sqrt{3}}{2})^2 = \frac{3}{4}$

$\boxed{\text{or}}$

$y = x^2 \Rightarrow y = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$ $(\frac{\sqrt{3}}{2}, \frac{3}{4})$ $(-\frac{\sqrt{3}}{2}, \frac{3}{4})$

$\hookrightarrow y = (\frac{-\sqrt{3}}{2})^2 = \frac{3}{4}$

- The tangent line for $y = x^2$

$\bar{y} = 2x \rightarrow \bar{y}\Big|_{(-\frac{\sqrt{3}}{2}, \frac{3}{4})} = 2(\frac{-\sqrt{3}}{2}) = -\sqrt{3}$

$y = -\sqrt{3}\, x + c \leftrightarrow$

$$\frac{3}{4} = \left(\frac{-\sqrt{3}}{2}\right)(-\sqrt{3}) + c \Rightarrow \frac{3}{4} = \frac{3}{2} + c$$

$$c = \frac{3}{4} - \frac{3}{2} \Rightarrow c = \frac{-3}{4}$$

$$y = -\sqrt{3}\,x - \frac{3}{4}$$

The tangent lines for $y = \frac{3}{2} - x^2$

$$y' = -2x \Rightarrow y'\Big|_{\left(\frac{-\sqrt{3}}{2}, \frac{3}{4}\right)} = -2\left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3}$$

$$y = \sqrt{3}\,x + c \Rightarrow$$

$$\frac{3}{4} = \sqrt{3}\left(\frac{-\sqrt{3}}{2}\right) + c \Rightarrow c = \frac{9}{4}$$

$$y = \sqrt{3}\,x + \frac{9}{4}$$

The corresponding normales are

$n_1 = \sqrt{3}\, i + j \quad , n_2 = -\sqrt{3}\, i + j$

- The angle at $\left(\frac{-\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$

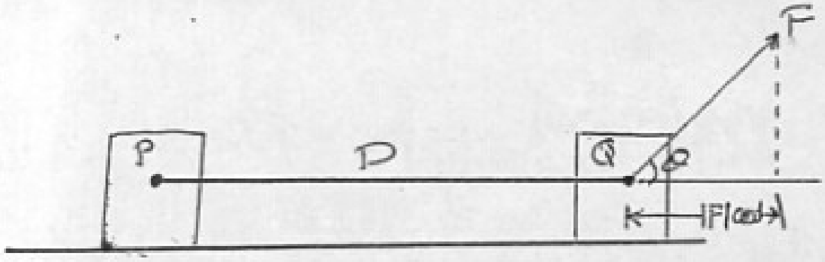
$$\theta = \cos^{-1}\left(\frac{-3+1}{(\sqrt{4})(\sqrt{4})}\right) = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

and the same method for the point $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$

Work

A work has been done by force F in moving an object through a distance d is $W = F\,d$.

- If a force F moving an object through a displacement $D = \overrightarrow{PQ}$ has some other direction, If θ is the angle between F and D



$$\text{Work} = \begin{pmatrix}\text{scalar component of} \\ F \text{ in the direction of} \\ D\end{pmatrix} (\text{length of } D)$$

$$= (|F| \cos\theta)\,|D|$$

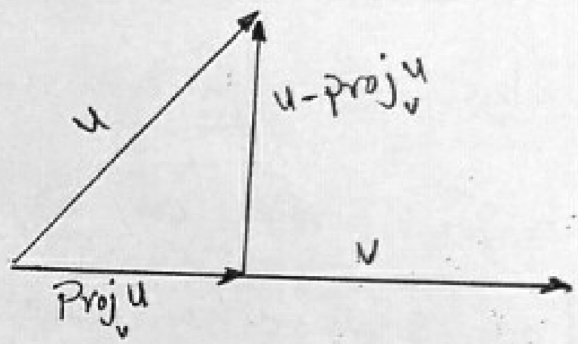
$$= F \cdot D$$

Ex If $|F| = 40 N$, $|D| = 3m$, and $\theta = 60°$, the work done by F in acting from P to Q is

$$W = |F||D| \cos\theta$$
$$= (40)(3) \cos 60 = 60 \quad N.m (J) \text{ (Joules)}$$

Writing a vector as a Sum of orthogonal Vectors

One way to write a vector $u = \langle a, b \rangle$ as a sum of orthogonal vectors $u = \langle a, b \rangle = ai + bj$ (Since $i \cdot j =$



but
Generally, for vectors u & v, it is easy to write v as $u - proj_v u$
is orthogonal to the projection vector $Proj_v u$

$$\therefore \quad u = Proj_v u + (u - proj_v u)$$

express u as a sum of orthogonal vectors

$$\therefore \quad u = Proj_v u + (u - proj_v u)$$

$$= \underbrace{\left(\frac{u \cdot v}{|v|^2}\right) v}_{\text{parallel to } v} + \underbrace{\left(u - \left(\frac{u \cdot v}{|v|^2}\right) v\right)}_{\text{orthogonal to } v}$$

<u>Ex</u>

The vector $v = 2i + 3j$ is tangent to the curve

$$y = \frac{x^2}{2} + \frac{1}{2} \quad \text{at } P = (1,1)$$

If $u = 4i - j$ is the acceleration.

Express u as the sum of vector parallel to v and vector orthogonal to v.

<u>Sol</u> $u \cdot v = 8 - 3 = 5$ and $|v|^2 = v \cdot v = 4 + 9 = 13$

$$u = \left(\frac{u \cdot v}{|v|^2}\right) v + \left(u - \left(\frac{u \cdot v}{|v|^2}\right) v\right)$$

$$= \frac{5}{13}(2i + 3j) + \left(4i - j - \frac{5}{13}(2i + 3j)\right)$$

$$= \left(\frac{10}{13} i + \frac{15}{13} j\right) + \left(\frac{42}{13} i - \frac{28}{13} j\right)$$