Al-Ayen University
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# Mechanics

## Dynamics

Title: Mechanical Vibration

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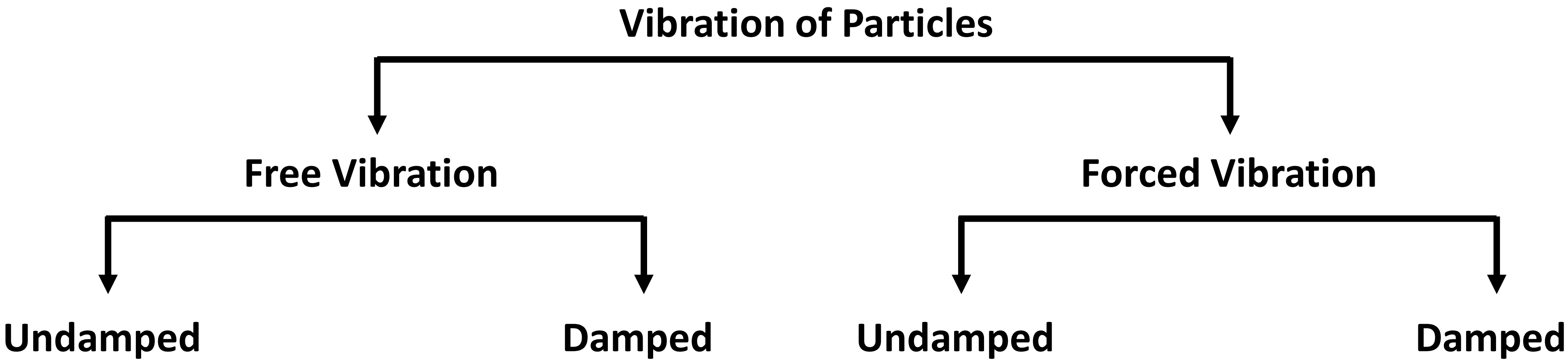
Lecture No. 11

## Vibration of Particles

When a spring-mounted body is disturbed from its equilibrium position, its ensuing motion in the absence of any imposed external forces is termed free vibration. In every actual case of free vibration, there exists some retarding or damping force which tends to diminish the motion. Common damping forces are those due to mechanical and fluid friction.

The general state of vibration of particles can be generally indicated from the following schematic representation.




**Note:** In case of a rigid body (instead of a particle), the main concerns are about the **translational** and **rotational** vibrations where each one of them can also be classified as shown in the above schematic representation.

## Equation of Motion for Undamped Free Vibration

We begin by considering the horizontal vibration of the simple frictionless spring-mass system of Figure below. Note that the variable x denotes the displacement of the mass from the equilibrium position, which, for this system, is also the position of zero spring deflection.

Applying Newton's second law in the form $\Sigma F_x = m\ddot{x}$ gives

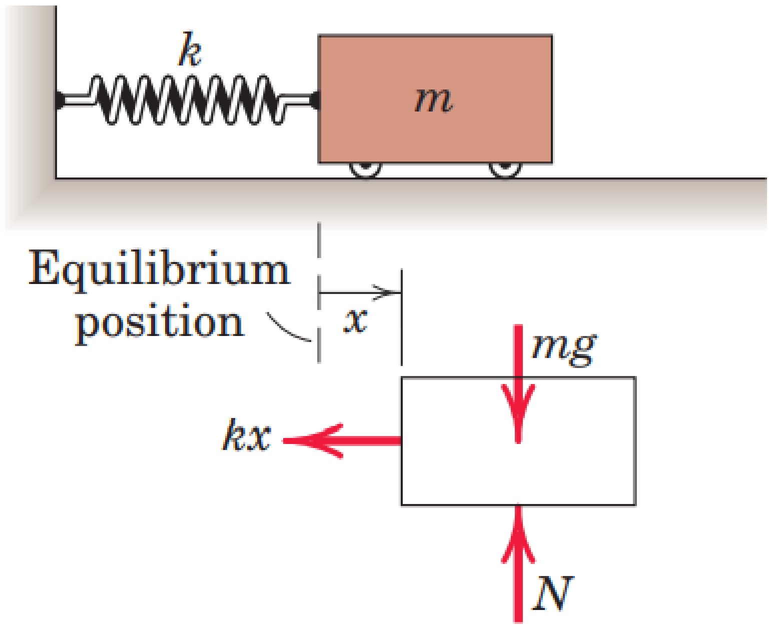
$$-kx = m\ddot{x} \qquad \text{or} \qquad m\ddot{x} + kx = 0$$

The above equation is normally written as

$$\ddot{x} + \omega_n{}^2 x = 0$$

where

$$\omega_n = \sqrt{k/m}$$



## Solution for Undamped Free Vibration

Because we anticipate an oscillatory motion, we look for a solution which gives x as a periodic function of time. Thus, a logical choice is

$$x = A\cos\omega_n t + B\sin\omega_n t \qquad (1)$$

or, alternatively

$$x = C\sin(\omega_n t + \psi) \qquad (2)$$

We determine the constants A and B or C and $\psi$, from knowledge of the initial displacement $\boldsymbol{x_o}$ and initial velocity $\boldsymbol{\dot{x}_o}$ of the mass. We obtain

$$x_0 = A \quad \text{and} \quad \dot{x}_0 = B\omega_n$$

Substitution of these values of A and B into Eq.1 yields

$$x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \qquad (3)$$

$$x_0 = C\sin\psi \quad \text{and} \quad \dot{x}_0 = C\omega_n\cos\psi$$

Solving for C and $\psi$ yields

$$C = \sqrt{{x_0}^2 + (\dot{x}_0/\omega_n)^2} \qquad \psi = \tan^{-1}(x_0\omega_n/\dot{x}_0)$$

Substitution of these values of C and $\psi$ into Eq.2 yields

$$x = \sqrt{{x_0}^2 + (\dot{x}_0/\omega_n)^2}\,\sin[\omega_n t + \tan^{-1}(x_0\omega_n/\dot{x}_0)] \qquad (4)$$

Equations 3 and 4 represent two different mathematical expressions for the same time-dependent motion.

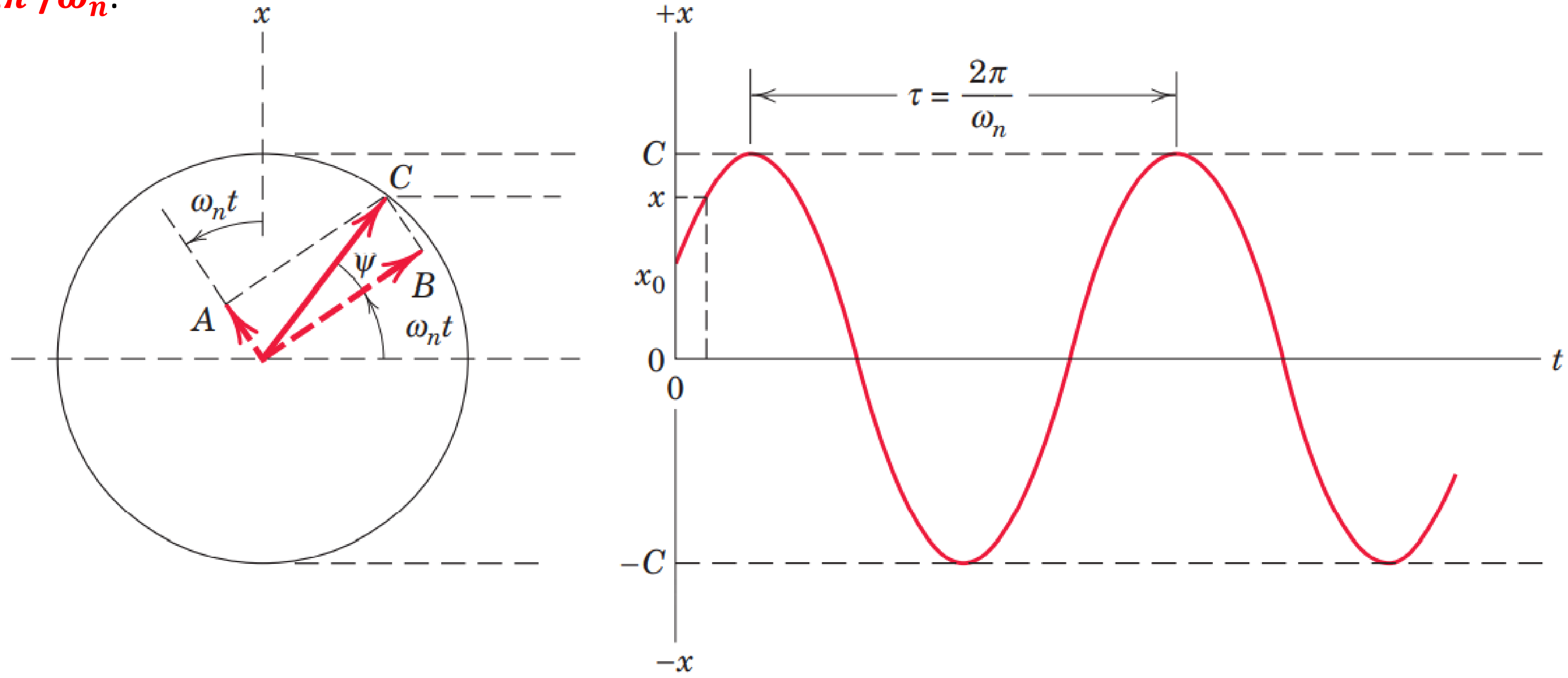
## Graphical Representation of Motion

The motion may be represented graphically, Figure below, where x is seen to be the projection onto a vertical axis of the rotating vector of length C.

The vector rotates at the constant angular velocity $\boldsymbol{\omega_n = \sqrt{k/m}}$, which is called the natural circular frequency and has the units radians per second.

The number of complete cycles per unit time is the natural frequency $\boldsymbol{f_n = \omega_n / 2\pi}$ and is expressed in hertz (1 hertz (Hz) = 1 cycle per second).

The time required for one complete motion cycle (one rotation of the reference vector) is the period of the motion and is given by $\boldsymbol{\tau = 1/f_n = 2\pi/\omega_n}$.



## Example

A body weighing 25 lb is suspended from a spring of constant $k = 160$ lb/ft. At time $t = 0$, it has a downward velocity of 2 ft/sec as it passes through the position of static equilibrium. Determine

(*a*) the static spring deflection $\delta_{st}$

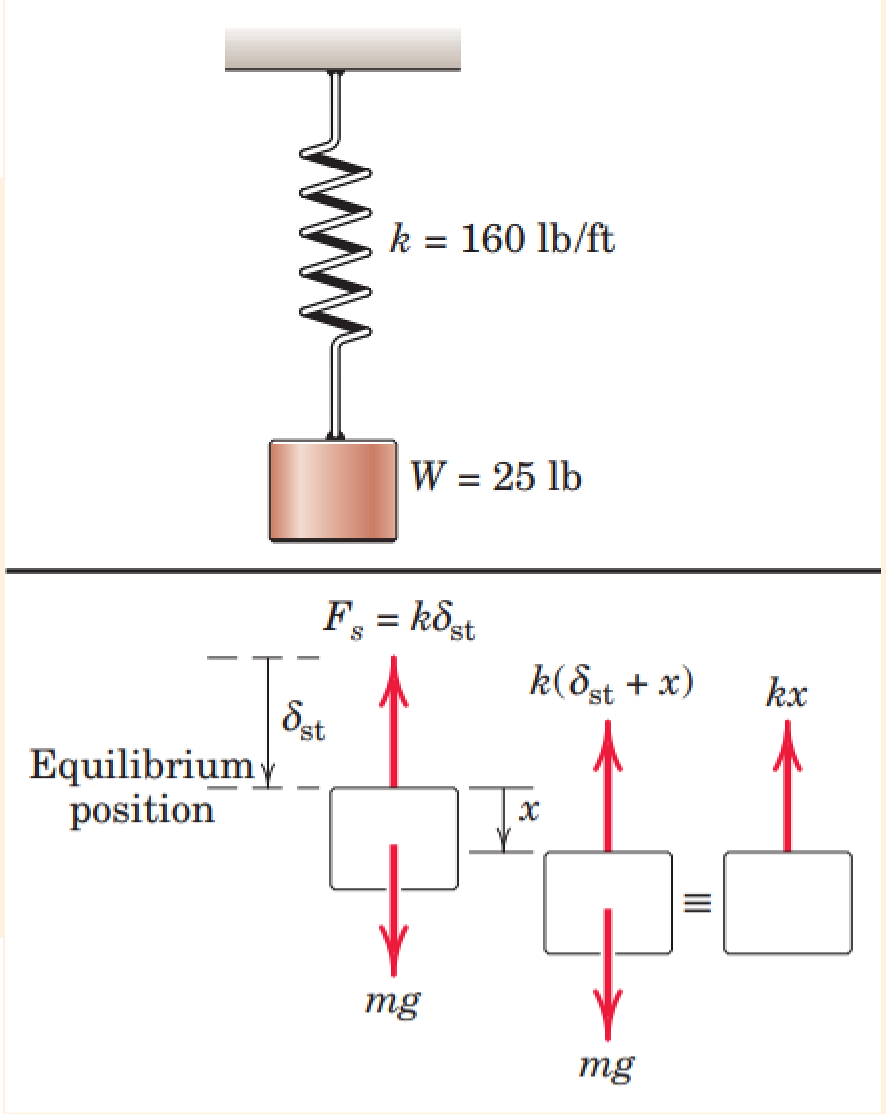
(*b*) the natural frequency of the system in both rad/sec ($\omega_n$) and cycles/sec ($f_n$)

(*c*) the system period $\tau$

(*d*) the displacement $x$ as a function of time, where $x$ is measured from the position of static equilibrium

(*e*) the maximum velocity $v_{max}$ attained by the mass

(*f*) the maximum acceleration $a_{max}$ attained by the mass.



**Solution.** (*a*) From the spring relationship $F_s = kx$, we see that at equilibrium

$$mg = k\delta_{st} \qquad \delta_{st} = \frac{mg}{k} = \frac{25}{160} = 0.1562 \text{ ft or } 1.875 \text{ in.} \qquad \textit{Ans.}$$

(*b*)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160}{25/32.2}} = 14.36 \text{ rad/sec} \qquad \textit{Ans.}$$

$$f_n = (14.36)\left(\frac{1}{2\pi}\right) = 2.28 \text{ cycles/sec} \qquad \textit{Ans.}$$

(*c*)

$$\tau = \frac{1}{f_n} = \frac{1}{2.28} = 0.438 \text{ sec} \qquad \textit{Ans.}$$

(*d*) From Eq. 8/6:

$$\begin{aligned} x &= x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \\ &= (0) \cos 14.36t + \frac{2}{14.36} \sin 14.36t \\ &= 0.1393 \sin 14.36t \text{ ft} \qquad \textit{Ans.} \end{aligned}$$

As an exercise, let us determine $x$ from the alternative Eq. 8/7:

$$\begin{aligned} x &= \sqrt{{x_0}^2 + (\dot{x}_0/\omega_n)^2} \sin \left[\omega_n t + \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)\right] \\ &= \sqrt{0^2 + \left(\frac{2}{14.36}\right)^2} \sin \left[14.36t + \tan^{-1}\left(\frac{(0)(14.36)}{2}\right)\right] \\ &= 0.1393 \sin 14.36t \text{ ft} \end{aligned}$$

(*e*) The velocity is $\dot{x} = 14.36(0.1393) \cos 14.36t = 2 \cos 14.36t$ ft/sec. Because the cosine function cannot be greater than 1 or less than −1, the maximum velocity $v_{max}$ is 2 ft/sec, which, in this case, is the initial velocity. *Ans.*

(*f*) The acceleration is

$$\ddot{x} = -14.36(2) \sin 14.36t = -28.7 \sin 14.36t \text{ ft/sec}^2$$

The maximum acceleration $a_{max}$ is 28.7 ft/sec$^2$. *Ans.*