Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

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L15: Incorporation of Neumann Boundary Conditions in the Block-Centered Grid system

Outline

- Incorporation of Boundary Conditions/Block-Centered Grid
  - Neumann Boundary Conditions
  - Exercise

Incorporation of Boundary Conditions/Block-Centered Grid

- Neumann Boundary Conditions (pressure gradients specified at the boundaries).
- For generality consider a heterogeneous reservoir with sources/sinks.

$$\text{PDE} \quad 1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

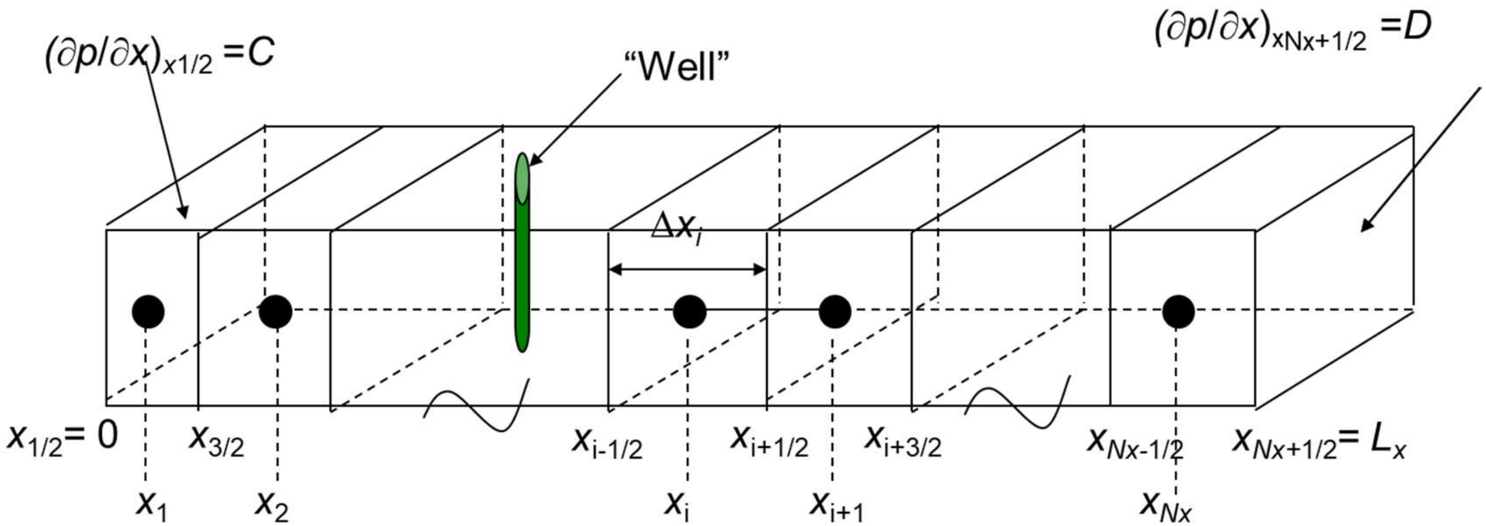
$$\text{IC} \quad p(x,t=0)=P_{Initial},\ 0\le x\le L_x,$$

$$\text{BC} \quad \frac{\partial p(x=0,t>0)}{\partial x}=C$$

$$\text{BC} \quad \frac{\partial p(x=L_x,t>0)}{\partial x}=D$$

Note: if C = 0 and/or D = 0, then we have no – flow (closed) boundary.

- Neumann BCs (pressure specified at the boundaries)



• Implicit Difference Equation: Multiply both sides by the bulk volume of the grid-block *i*, $V_{b,i} = \Delta x_i wh$

$$1.127\times10^{-3}\Delta x_i wh\left[\frac{\lambda_{x,i+1/2}\left(\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-q_{sc,i}^{\ n+1}B=\frac{(\phi c_t)_i\Delta x_i wh}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

$$T_{x,i+1/2}\left(p_{i+1}^{n+1}-p_i^{n+1}\right)-T_{x,i-1/2}\left(p_i^{n+1}-p_{i-1}^{n+1}\right)-q_{sc,i}^{\ n+1}B=\widetilde{V}_i\left(p_i^{n+1}-p_i^n\right)$$

or

$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$

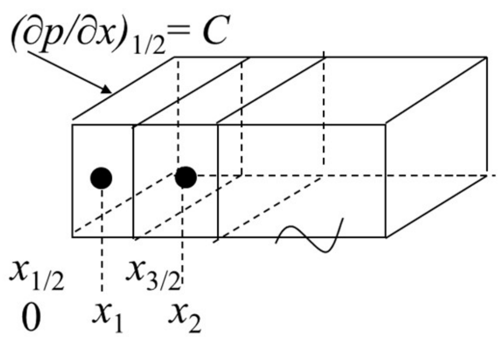
$$T_{x,i+1/2}=1.127\times10^{-3}\frac{\lambda_{x,i+1/2}wh}{\left(x_{i+1}-x_i\right)},\ T_{x,i-1/2}=1.127\times10^{-3}\frac{\lambda_{x,i-1/2}wh}{\left(x_i-x_{i-1}\right)},\ and\ \widetilde{V}_i=\frac{(\phi c_t)\Delta x_i wh}{5.615\Delta t^{n+1}}$$

- Recall the discretized PDE for *i* =1

$$1.127\times10^{-3}\Delta x_i wh\left[\frac{\lambda_{x,i+1/2}\left(\dfrac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\dfrac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-q_{sc,i}^{\;n+1}B=\frac{(\phi c_t)_i\Delta x_i wh}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

$$1.127\times10^{-3}\Delta x_1 wh\left[\frac{\lambda_{x,3/2}\left(\dfrac{p_2^{n+1}-p_1^{n+1}}{x_2-x_1}\right)-\lambda_{x,1/2}\left(\dfrac{\partial p}{\partial x}\right)_{1/2}^{n+1}}{x_{3/2}-x_{1/2}}\right]-q_{sc,1}^{\;n+1}B=\frac{(\phi c_t)_1\Delta x_1 wh}{5.615}\left(\frac{p_1^{n+1}-p_1^n}{\Delta t^{n+1}}\right)$$

$= C$



- Recall for i =1,

$$1.127\times10^{-3}\Delta x_1 wh\left[\frac{\lambda_{x,3/2}\left(\dfrac{p_2^{n+1}-p_1^{n+1}}{x_2-x_1}\right)-\lambda_{x,1/2}\left(\dfrac{\partial p}{\partial x}\right)_{1/2}^{n+1}}{x_{3/2}-x_{1/2}}\right]-q_{sc,1}^{n+1}B=\frac{(\phi c_t)_1\Delta x_1 wh}{5.615}\left(\frac{p_1^{n+1}-p_1^n}{\Delta t^{n+1}}\right)$$

where $\left(\dfrac{\partial p}{\partial x}\right)_{1/2}^{n+1} = C$

$$\left(T_{x,3/2}+\widetilde{V}_1\right)p_1^{n+1}-T_{x,3/2}p_2^{n+1}=-q_{sc,1}^{n+1}B+\widetilde{V}_1 p_1^n+\widetilde{T}_{x,1/2}C$$

where we evaluate $\widetilde{T}_{x,1/2}$ and $\widetilde{V}_1$ from:

$$\widetilde{T}_{x,1/2}=1.127\times10^{-3}\lambda_{x,1/2}wh \quad \text{and} \quad \widetilde{V}_1=\frac{(\phi c_t)\Delta x_1 wh}{5.615\Delta t^{n+1}}$$

if we have no-flow left boundary (i.e., C = 0),

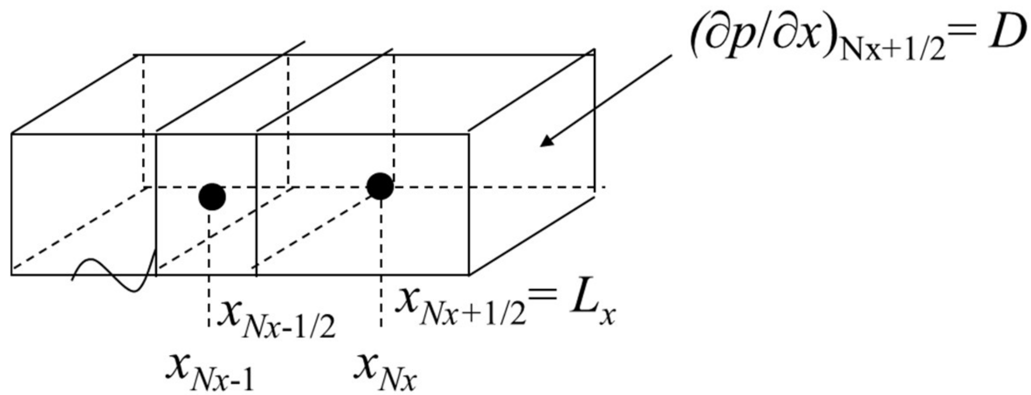
$$\left(T_{x,3/2}+\widetilde{V}_1\right)p_1^{n+1}-T_{x,3/2}p_2^{n+1}=-q_{sc,1}^{n+1}B+\widetilde{V}_1 p_1^n+\widetilde{T}_{x,1/2}C$$

where the last term $\widetilde{T}_{x,1/2}C = 0$

- Recall the discretized PDE for $i = N_x$

$$= D$$

$$1.127\times10^{-3}\Delta x_{Nx} wh\left[\frac{\lambda_{x,Nx+1/2}\left(\frac{\partial p}{\partial x}\right)_{Nx+1/2}^{n+1} - \lambda_{x,Nx-1/2}\left(\frac{p_{Nx}^{n+1}-p_{Nx-1}^{n+1}}{x_{Nx}-x_{Nx-1}}\right)}{x_{Nx+1/2}-x_{Nx-1/2}}\right] - q_{sc,Nx}^{n+1}B = \frac{(\phi c_t)_{Nx}\Delta x_{Nx} wh}{5.615}\left(\frac{p_{Nx}^{n+1}-p_{Nx}^{n}}{\Delta t^{n+1}}\right)$$



• Recall the discretized PDE for $i = N_x$

$$1.127\times10^{-3}\Delta x_{Nx} wh\left[\frac{\lambda_{x,Nx+1/2}\left(\frac{\partial p}{\partial x}\right)_{Nx+1/2}^{n+1} - \lambda_{x,Nx-1/2}\left(\frac{p_{Nx}^{n+1}-p_{Nx-1}^{n+1}}{x_{Nx}-x_{Nx-1}}\right)}{x_{Nx+1/2}-x_{Nx-1/2}}\right] - q_{sc,Nx}^{n+1}B = \frac{(\phi c_t)_{Nx}\Delta x_{Nx} wh}{5.615}\left(\frac{p_{Nx}^{n+1}-p_{Nx}^{n}}{\Delta t^{n+1}}\right)$$

with $\left(\frac{\partial p}{\partial x}\right)_{Nx+1/2}^{n+1} = D$

$$-T_{x,Nx-1/2}p_{Nx-1}^{n+1} + \left(T_{x,Nx-1/2}+\widetilde{V}_{Nx}\right)p_{Nx}^{n+1} = -q_{sc,Nx}^{n+1}B + \widetilde{V}_{Nx}p_{Nx}^{n} + \widetilde{T}_{x,Nx+1/2}D$$

where we evaluate $\widetilde{T}_{x,Nx+1/2}$ and $\widetilde{V}_{Nx}$ from:

$$\widetilde{T}_{x,Nx+1/2} = 1.127\times10^{-3}\lambda_{x,Nx+1/2}wh$$

$$\widetilde{V}_{Nx} = \frac{(\phi c_t)\Delta x_{Nx} wh}{5.615\Delta t^{n+1}}$$

if we have no-flow right boundary (i.e., $D = 0$),

$$-T_{x,Nx-1/2}p_{Nx-1}^{n+1} + \left(T_{x,Nx-1/2}+\widetilde{V}_{Nx}\right)p_{Nx}^{n+1} = -q_{sc,Nx}^{n+1}B + \widetilde{V}_{Nx}p_{Nx}^{n} + \widetilde{T}_{x,Nx+1/2}D$$

with $\widetilde{T}_{x,Nx+1/2}D \rightarrow 0$

Exercise

Fig. 1 shows a closed-boundary reservoir model with 1-D block-centered grid. Determine the pressure distribution after 20 Days of production. The initial reservoir pressure is 6000 psia. The rock and fluid properties for this problem are $c_t = 3.5\times10^{-6}$ psi^{-1}, $\varphi = 0.18$, B = 1 RB/STB, $\mu = 10$ cp. Use the explicit approximation method with a time-step size of $\Delta t = 2$ Days.

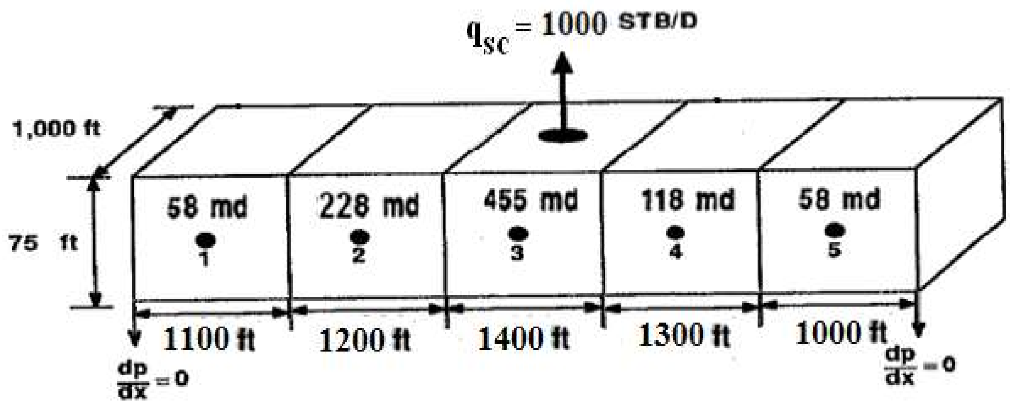


Figure 1: A closed-boundary reservoir model with 1-D block-centered grid

Solution:

1. To find the pressure distribution, we need to determine the partial differential equation (PDE) that can be used to model this reservoir system. Since the system is: 1-D, linear heterogeneous reservoir, single phase flow of slightly compressible fluid with Source and Sink, the suitable PDE is:

$$1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t}$$

2. Approximation of this PDE by explicit finite differences is:

$$1.127\times10^{-3}\left[\frac{\lambda_{x,i+1/2}\left(\dfrac{p_{i+1}^n-p_i^n}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\dfrac{p_i^n-p_{i-1}^n}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-\frac{q_{sc,i}^{n+1}B}{\Delta x_i wh}=\frac{(\phi c_t)_i}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

or:

$$T_{x,i+1/2}\left(p_{i+1}^n-p_i^n\right)-T_{x,i-1/2}\left(p_i^n-p_{i-1}^n\right)-q_{sc,i}^{n+1}B=\widetilde{V}_i\left(p_i^{n+1}-p_i^n\right) \quad \text{...............(1)}$$

$$T_{x,i+1/2}\left(p_{i+1}^{n} - p_{i}^{n}\right) - T_{x,i-1/2}\left(p_{i}^{n} - p_{i-1}^{n}\right) - q_{sc,i}^{\;n+1} B = \widetilde{V}_i\left(p_i^{n+1} - p_i^{n}\right) \quad \text{......................................(1)}$$

Where: $\lambda_{x,i\mp1/2} = \dfrac{k_{x,i\mp1/2}}{\mu_{i\mp1/2}}$

$$k_{x,i\mp1/2} = \frac{k_{x,i\mp1}\, k_{x,i}\, (\Delta x_{i\mp1} + \Delta x_i)}{k_{x,i}\, \Delta x_{i\mp1} + k_{x,i\mp1}\, \Delta x_i} \quad \text{......................................(2)}$$

$$T_{x,i+1/2} = 1.127\times10^{-3}\,\frac{\lambda_{x,i+1/2} wh}{\left(x_{i+1} - x_i\right)} = 2\times1.127\times10^{-3}\,\frac{\lambda_{x,i+1/2} wh}{\left(\Delta x_{i+1} + \Delta x_i\right)} \quad \text{......................................(3)}$$

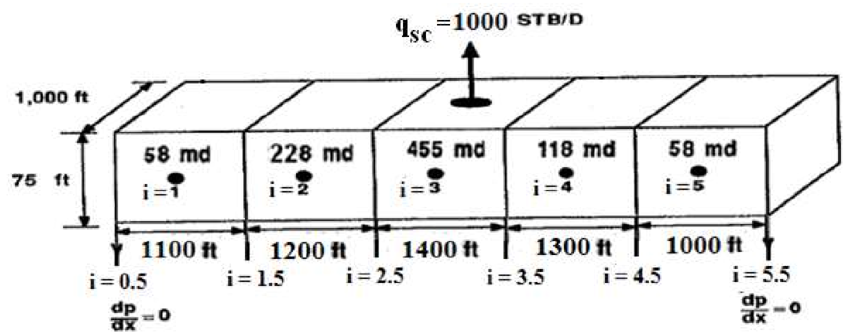
$$T_{x,i-1/2} = 1.127\times10^{-3}\,\frac{\lambda_{x,i-1/2} wh}{\left(x_i - x_{i-1}\right)} = 2\times1.127\times10^{-3}\,\frac{\lambda_{x,i-1/2} wh}{\left(\Delta x_i + \Delta x_{i-1}\right)}$$

$$\widetilde{V}_i = \frac{\left(\phi c_t\right)_i \Delta x_i wh}{5.615\Delta t^{n+1}} \quad \text{......................................(4)}$$

P_i^{n+1} can be obtained from Eq. 1:

$$p_i^{n+1} = p_i^n + (1/\widetilde{V}_i)\left[T_{x,i+1/2}(p_{i+1}^n - p_i^n) - T_{x,i-1/2}(p_i^n - p_{i-1}^n) - q_{sc,i}^{n+1}B\right]$$ ……………………………………(5)

3. Find transmissibility at the boundaries of blocks:



i	K_i (Eq. 2), md	K_i/μ, md/cp	T_i (Eq. 3), B/(D.psi)
0.5 (Closed Boundary)	-	-	0
1.5	94.92884	9.49288	0.69773
2.5	311.74757	31.17476	2.02696
3.5	191.57262	19.15726	1.19946
4.5	81.39193	8.13919	0.59823
5.5(Closed Boundary)	-	-	0

4. Calculate $\tilde{V}_i$ from Eq. 4:

i	$\tilde{V}_i$, B/(D.psi)
1	4.62823
2	5.04898
3	5.89047
4	5.46972
5	4.20748

5. Calculate the pressures from Eq. 5:

$$p_i^{n+1} = p_i^n + (1/\tilde{V}_i)\left[T_{x,i+1/2}\left(p_{i+1}^n - p_i^n\right) - T_{x,i-1/2}\left(p_i^n - p_{i-1}^n\right) - q_{sc,i}^{n+1} B\right] \qquad \text{.................(5)}$$

For i=1:

$$P_1^{n+1} = P_1^n + \left(\frac{1}{4.62823}\right)[0.69773(P_2^n - P_1^n) - 0]$$

For i=2:

$$P_2^{n+1} = P_2^n + \left(\frac{1}{5.04898}\right)[2.02696(P_3^n - P_2^n) - 0.69773(P_2^n - P_1^n) - 0]$$

For i=3:

$$P_3^{n+1} = P_3^n + \left(\frac{1}{5.89047}\right)[1.19946(P_4^n - P_3^n) - 2.02696(P_3^n - P_2^n) - 1000 \times 1]$$

For i=4:

$$P_4^{n+1} = P_4^n + \left(\frac{1}{5.46972}\right)[0.59823(P_5^n - P_4^n) - 1.19946(P_4^n - P_3^n) - 0]$$

For i=5:

$$P_5^{n+1} = P_5^n + \left(\frac{1}{4.20748}\right)[0 - 0.59823(P_5^n - P_4^n) - 0]$$

The results are shown in the Table below:

n	t=nxΔt, Days	P1, psia	P2, psia	P3, psia	P4, psia	P5, psia
0	0	6000	6000	6000	6000	6000
1	2	6000	6000	5830.234316	6000	6000
2	4	6000	5931.845974	5753.455197	5962.77203	6000
3	6	5989.72546	5869.64765	5687.697712	5920.94255	5994.706828
4	8	5971.62317	5813.195908	5628.037344	5877.86191	5984.218826
5	10	5947.73952	5760.755611	5572.857165	5834.7102	5969.096717
6	12					
7	14					
8	16					
9	18					
10	20					

THANK YOU