Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

Lecturer: Dr. Mohammed Idrees Al-Mossawy

2020/2021

L13: Simulation of 1D Linear Flow with Sources/Sinks

Outline

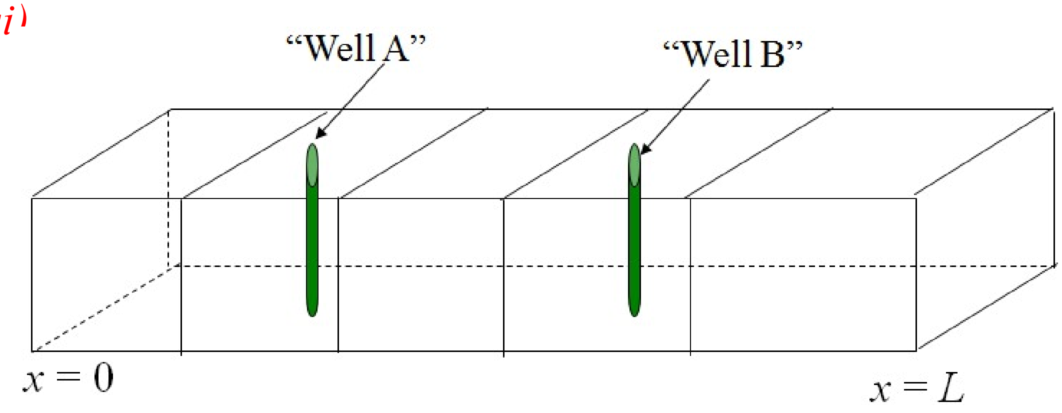
- Simple 1D Linear Flow in a *Homogeneous* Linear-Reservoir with Sources/Sinks
  - Exercise
- Simple 1D Linear Flow in a *Heterogeneous* Linear-Reservoir with Sources/Sinks
- Computation of Transmissibility
- Harmonic Average Permeability
  - Example

Simple 1D Linear Flow in a Homogeneous Linear-Reservoir with Sources/Sinks

- 1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir with sources/sinks.
- For this case, it is convenient to include a source/sink term to model injection/production from a well or wells.

$$\text{PDE} \quad 1.127\times10^{-3}\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2}-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t}\ , \quad 0<x<L, t>0$$

q_{sc} = surface rate (STB/D), q_{sc} > 0 for production (sink) well, and q_{sc} < 0 for source (injection) well, V_b = bulk volume (cuft), k (md), μ (cp), p (psi), t (days), x (ft), c_t(1/psi)



Simple 1D Linear Flow in a Homogeneous Linear-Reservoir with Sources/Sinks

$$\text{PDE} \quad 1.127\times10^{-3}\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2}-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t} \;, \quad 0<x<L, t>0$$

We evaluate both sides of the PDE at x_i,t^{n+1}. So, approximate the second derivative at x_i,t^{n+1} as:

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_{x_i,t^{n+1}} \approx \frac{p_{i+1}^{n+1}-2p_i^{n+1}+p_{i-1}^{n+1}}{(\Delta x)^2}$$

Approximate the source/sink term at x_i,t^{n+1} as:

$$\frac{q_{sc,i}^{n+1}B}{V_{b,i}} \qquad (V_{b,i}=wh\Delta x)$$

Approximate the time derivative at x_i,t^{n+1} as:

$$\left(\frac{\partial p}{\partial t}\right)_{x_i,t^{n+1}} \approx \frac{p_i^{n+1}-p_i^n}{\Delta t}$$

Simple 1D Linear Flow in a Homogeneous Linear-Reservoir with Sources/Sinks

$$\text{PDE} \quad 1.127\times10^{-3}\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} - \frac{q_{sc}(x,t)B}{V_b} = \frac{\phi c_t}{5.615}\frac{\partial p}{\partial t} \;,\quad 0 < x < L, t > 0$$

Substitute the finite difference approximation in PDE gives:

$$1.127\times10^{-3}\frac{k}{\mu}\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{(\Delta x)^2} - \frac{q_{sc,i}^{n+1}B}{wh\Delta x} = \frac{\phi c_t}{5.615}\left(\frac{p_i^{n+1} - p_i^n}{\Delta t}\right)$$

The above equation can be rearranged as (after multiplying both sides by the bulk volume, $wh\Delta x$):

$$-T\, p_{i-1}^{n+1} + (\widetilde{V} + 2T)\, p_i^{n+1} - T\, p_{i+1}^{n+1} = -q_{sc,i}^{n+1}\, B + \widetilde{V} p_i^n$$

$$T = \text{Transmissibility} = 1.127\times10^{-3}\frac{kwh}{\mu\Delta x} \qquad \widetilde{V} = \frac{\phi c_t wh\Delta x}{5.615\Delta t}$$

Note: This equation and terms (T and $\widetilde{V}$) are only valid for 1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir with sources/sinks.

Simple 1D Linear Flow in a Homogeneous Linear-Reservoir with Sources/Sinks

Matrix Problem:

$$\underbrace{\begin{bmatrix} (\tilde{V}+2T) & -T & 0 & 0 & \cdots & 0 \\ -T & (\tilde{V}+2T) & -T & 0 & \cdots & 0 \\ 0 & -T & (\tilde{V}+2T) & -T & 0 & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -T & (\tilde{V}+2T) & -T \\ 0 & \cdots & \cdots & 0 & -T & (\tilde{V}+2T) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p_1^{n+1} \\ p_2^{n+1} \\ \vdots \\ p_i^{n+1} \\ \vdots \\ p_{Nx-1}^{n+1} \end{bmatrix}}_{\vec{p}^{\,n+1}} = \underbrace{\begin{bmatrix} \tilde{V}p_1^n + Tp_0^{n+1} - q_{sc,1}^{n+1}B \\ \tilde{V}p_2^n - q_{sc,2}^{n+1}B \\ \vdots \\ \tilde{V}p_i^n - q_{sc,i}^{n+1}B \\ \vdots \\ \tilde{V}p_{Nx-1}^n + Tp_{Nx}^{n+1} - q_{sc,Nx-1}^{n+1}B \end{bmatrix}}_{\vec{d}^{\,n}}$$

A is a symmetric, tri-diagonal matrix can be solved by ***direct methods*** or ***iterative methods***

Note : We will set q_{sc}s to zero in the right-hand side vector if the grid point does not contain a source/sink.

Exercise

Use the explicit finite difference method to derive an equation to calculate the pressure in 1D single phase flow of slightly compressible fluid in a homogeneous linear-reservoir with sources/sinks.

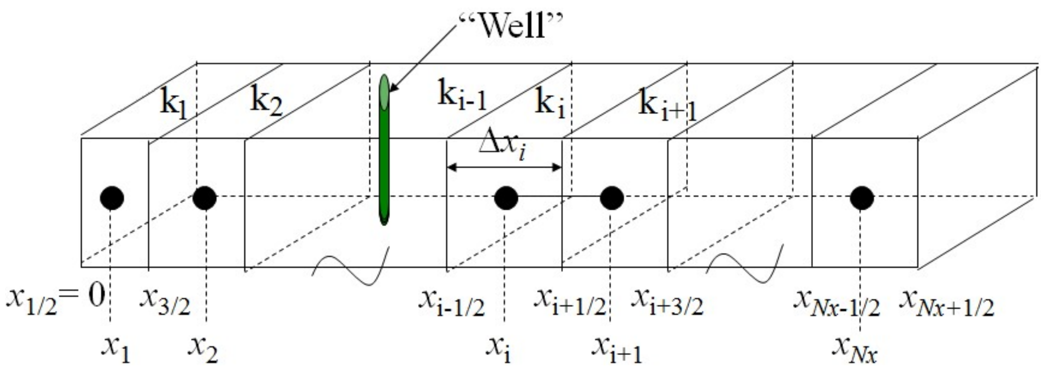
Simple 1D Linear Flow in a Heterogeneous Linear-Reservoir with Sources/Sinks

- 1D single phase flow of slightly compressible fluid in a heterogeneous linear-reservoir with sources/sinks.

$$\text{PDE} \quad 1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

q_{sc} = surface rate (STB/D), $q_{sc} > 0$ for production (sink) well, and $q_{sc} < 0$ for source (injection) well, V_b = bulk volume (cuft), k (md), μ (cp), p (psi), t (days), x (ft), c_t(1/psi).

- For heterogeneous reservoirs we may use grid-blocks with different sizes to approximate changes of permeability.



Simple 1D Linear Flow in a Heterogeneous Linear-Reservoir with Sources/Sinks

PDE $$1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

- Implicit Difference Equations-Spatial Derivative:

$$\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial p}{\partial x}\right)_{x_i,t^{n+1}}=\frac{\lambda_{x,i+1/2}\left(\frac{\partial p}{\partial x}\right)_{i+1/2}^{n+1}-\lambda_{x,i-1/2}\left(\frac{\partial p}{\partial x}\right)_{i-1/2}^{n+1}}{x_{i+1/2}-x_{i-1/2}}\quad \text{where } \lambda_x=k_x/\mu$$

$$\left(\frac{\partial p}{\partial x}\right)_{i+1/2}^{n+1}=\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}$$

$$\left(\frac{\partial p}{\partial x}\right)_{i-1/2}^{n+1}=\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}$$

$$\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial p}{\partial x}\right)_{x_i,t^{n+1}}=\frac{\lambda_{x,i+1/2}\left(\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}$$

Simple 1D Linear Flow in a Heterogeneous Linear-Reservoir with Sources/Sinks

PDE $$1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

- Implicit Difference Equations-Time Derivative

$$\left(\frac{\partial p}{\partial t}\right)_{x_i,t^{n+1}}=\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right),\ \ where\ \ \Delta t^{n+1}=t^{n+1}-t^n$$

- Implicit Difference Equation

$$1.127\times10^{-3}\left[\frac{\lambda_{x,i+1/2}\left(\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-\frac{q_{sc,i}^{\ n+1}B}{\Delta x_i wh}=\frac{(\phi c_t)_i}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

Simple 1D Linear Flow in a Heterogeneous Linear-Reservoir with Sources/Sinks

- Implicit Difference Equation: Multiply both sides by the bulk volume of the grid-block *i*, $V_{b,i} = \Delta x_i wh$

$$1.127\times10^{-3}\Delta x_i wh\left[\frac{\lambda_{x,i+1/2}\left(\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-q_{sc,i}^{\;n+1}B=\frac{(\phi c_t)_i\Delta x_i wh}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

$$T_{x,i+1/2}\left(p_{i+1}^{n+1}-p_i^{n+1}\right)-T_{x,i-1/2}\left(p_i^{n+1}-p_{i-1}^{n+1}\right)-q_{sc,i}^{\;n+1}B=\widetilde{V}_i\left(p_i^{n+1}-p_i^n\right)$$

or

$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$

$$T_{x,i+1/2}=1.127\times10^{-3}\frac{\lambda_{x,i+1/2}wh}{\left(x_{i+1}-x_i\right)},\quad T_{x,i-1/2}=1.127\times10^{-3}\frac{\lambda_{x,i-1/2}wh}{\left(x_i-x_{i-1}\right)},\quad and\quad \widetilde{V}_i=\frac{(\phi c_t)_i\Delta x_i wh}{5.615\Delta t^{n+1}}$$

Simple 1D Linear Flow in a Heterogeneous Linear-Reservoir with Sources/Sinks

- Transmissibility, $T_{x,i+1/2}$ and $T_{x,i-1/2}$:

$$T_{x,i+1/2} = 1.127\times 10^{-3}\frac{\lambda_{x,i+1/2}wh}{\left(x_{i+1}-x_i\right)} = 2\times 1.127\times 10^{-3}\frac{\lambda_{x,i+1/2}wh}{\left(\Delta x_{i+1}+\Delta x_i\right)}$$

$$T_{x,i-1/2} = 1.127\times 10^{-3}\frac{\lambda_{x,i-1/2}wh}{\left(x_i-x_{i-1}\right)} = 2\times 1.127\times 10^{-3}\frac{\lambda_{x,i-1/2}wh}{\left(\Delta x_i+\Delta x_{i-1}\right)}$$

where

$$\lambda_{x,i\mp 1/2} = \frac{k_{x,i\mp 1/2}}{\mu_{i\mp 1/2}}$$

Computation of Transmissibility

- As we saw already, the flow between the blocks are achieved by the transmissibility term (T_x).
- The transmissibility contain terms describing the permeability "between" blocks, fluid mobility (relative permeability and viscosity in the case of multi-phase flow).
- For a single-phase flow, for example the transmissibility in the x-direction is to be computed as follows:

Computation of Transmissibility

- We use harmonic averaging to compute the absolute permeability (or mobility for single-phase flow) at the block-to-block interface. This is based on the steady-state flow between the blocks and is the conventional approach in simulation.

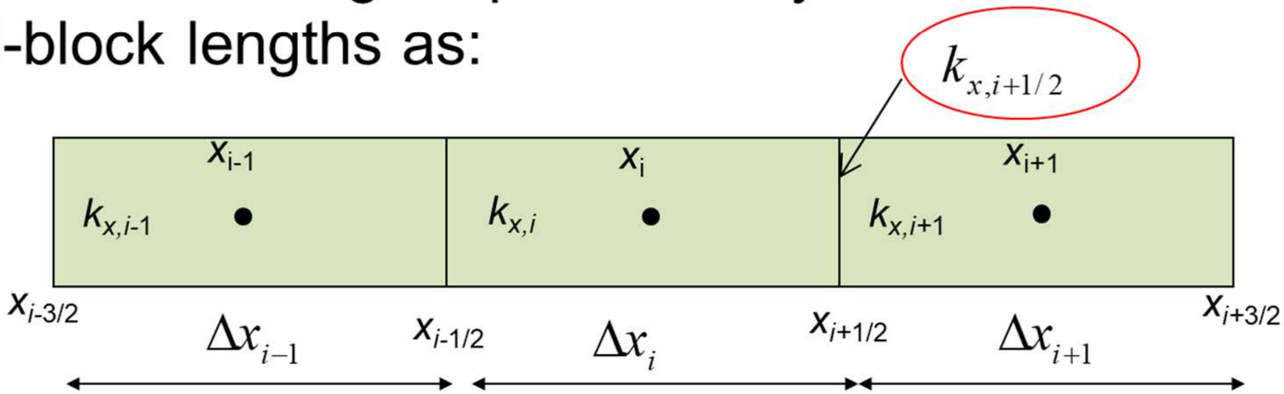
$$T_{x,i+1/2} = 1.127 \times 10^{-3} \frac{\lambda_{x,i+1/2}\, wh}{\left(x_{i+1} - x_i\right)}$$

$$\lambda_{x,i+1/2} = \frac{k_{x,i+1/2}}{\mu_{i+1/2}}$$

$$k_{x,i+1/2} = \frac{k_{x,i+1} k_{x,i} (x_{i+1} - x_i)}{k_{x,i} (x_{i+1} - x_{i+1/2}) + k_{x,i+1} (x_{i+1/2} - x_i)}$$

Harmonic Average Permeability

• For a block-centered grid, we can express the harmonic averaged permeability in terms of the grid-block lengths as:



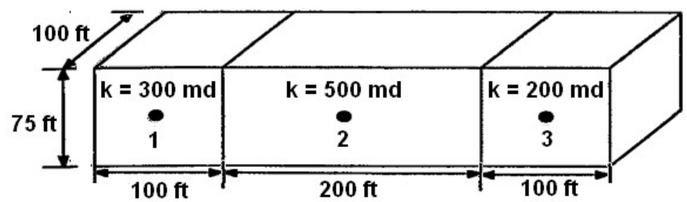
$$k_{x,i+1/2} = \frac{k_{x,i+1}\, k_{x,i}\, (x_{i+1} - x_i)}{k_{x,i}\, (x_{i+1} - x_{i+1/2}) + k_{x,i+1}\, (x_{i+1/2} - x_i)}$$

$$= \frac{k_{x,i+1}\, k_{x,i}\, (\Delta x_{i+1} + \Delta x_i)}{k_{x,i}\, \Delta x_{i+1} + k_{x,i+1}\, \Delta x_i}$$

Example

A 1-D block-centered grid system is shown in FIGURE. Assume a single-phase flow of slightly compressible fluid with following data for the system: k1 = 300 md, k2 = 500 md, k3 = 200 md, Bo = 1 RB/STB, ct = 3.5x10-6 psi-1, φ = 0.18, and μ = 10 cp. Calculate the transmissibility at the boundary between blocks 1 and 2. The following partial differential equation governs the flow system in this reservoir.

$$1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B_o}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t}$$



Solution

$$k_{x,i+1/2}=\frac{k_{x,i+1}\,k_{x,i}\,(\Delta x_{i+1}+\Delta x_i)}{k_{x,i}\,\Delta x_{i+1}+k_{x,i+1}\,\Delta x_i}$$

$$k_{x,1.5}=\frac{500\times300(200+100)}{300\times200+500\times100}=409.09\,md$$

$$T_{x,1.5}=2\times1.127\times10^{-3}\frac{(\frac{k}{\mu})_{x,i+1/2}wh}{(\Delta x_{i+1}+\Delta x_i)}=2\times1.127\times10^{-3}\times\frac{(\frac{409.09}{10})100\times75}{(200+100)}=2.305$$

THANK YOU