Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

Lecturer: Dr. Mohammed Idrees Al-Mossawy

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L12: Methods of Solving Systems of Linear Equations (Part 5)

(Matrices and Determinants-Part 2)

Outline

- Minor and Cofactor of Element
- Singular and Non-Singular Matrices
- Adjoint of a Matrix
- Inverse of a Matrix
- Solution of Linear Equations by Matrices
- Homework

Minor and Cofactor of Element:

For the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij}

Note: The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as

$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ Or $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors.

Example : Find the determinant of the matrix $A=\begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Solution

Step 1: Find the minors of the matrix.

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11}=\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}=(-1)(1)-(0)(2)=-1-0=-1$$

$$M_{12}=\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}=(3)(1)-(4)(2)=3-8=-5$$

$$M_{13}=\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}=(3)(0)-(4)(-1)=0+4=4$$

$$M_{21}=\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}=(2)(1)-(0)(1)=2-0=2$$

$$M_{22}=\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix}=(0)(1)-(4)(1)=0-4=-4$$

$$M_{23}=\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix}=(0)(0)-(4)(2)=0-8=-8$$

$$M_{31}=\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}=(2)(2)-(-1)(1)=4+1=5$$

$$M_{32}=\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}=(0)(2)-(3)(1)=0-3=-3$$

$$M_{33}=\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix}=(0)(-1)-(3)(2)=0-6=-6$$

$$\begin{array}{lll} M_{11}=-1 & M_{12}=-5 & M_{13}=4 \\ M_{21}=2 & M_{22}=-4 & M_{23}=-8 \\ M_{31}=5 & M_{32}=-3 & M_{33}=-6 \end{array}$$

Step 2: Determine the cofactors using the formula

$$C_{ij}=(-1)^{i+j}M_{ij}$$

$$C_{11}=(-1)^{1+1}M_{11}=(-1)^2(-1)=(1)(-1)=-1$$

$$C_{12}=(-1)^{1+2}M_{12}=(-1)^3(-5)=(-1)(-5)=5$$

$$C_{13}=(-1)^{1+3}M_{13}=(-1)^4(4)=(1)(4)=4$$

$$C_{21}=(-1)^{2+1}M_{21}=(-1)^3(2)=(-1)(2)=-2$$

$$C_{22}=(-1)^{2+2}M_{22}=(-1)^4(-4)=(1)(-4)=-4$$

$$C_{23}=(-1)^{2+3}M_{23}=(-1)^5(-8)=(-1)(-8)=8$$

$$C_{31}=(-1)^{3+1}M_{31}=(-1)^4(5)=(1)(5)=5$$

$$C_{32}=(-1)^{3+2}M_{32}=(-1)^5(-3)=(-1)(-3)=3$$

$$C_{33}=(-1)^{3+3}M_{33}=(-1)^6(-6)=(1)(-6)=-6$$

$$\begin{array}{lll} C_{11}=-1 & C_{12}=5 & C_{13}=4 \\ C_{21}=-2 & C_{22}=-4 & C_{23}=8 \\ C_{31}=5 & C_{32}=3 & C_{33}=-6 \end{array}$$

Step 3: Select *any* row or column of the matrix and then, using the formula

$|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$

$$|A|=a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}$$
$$=(3)(-2)+(-1)(-4)+(2)(8)=14$$

$$|A|=a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33}$$
$$=(1)(4)+(2)(8)+(1)(-6)=14$$

Determinant of a 4x4 matrix

Example : Find the determinant of the matrix $A=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

Solution

For this example, the second column will be used even though the third column is the simplest to solve for.

$$|A| = 2C_{12} + 1C_{22} + 2C_{32} + 4C_{42}$$

$$C_{12} = (-1)^3 \begin{vmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -\left[2(-1)^4 \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + 3(-1)^5 \begin{vmatrix} -1 & 0 \\ 3 & 0 \end{vmatrix} + 2(-1)^6 \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix}\right]$$

$$= -[2(0-0) - 3(0-0) + 2(0-0)] = 0$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} + 0(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 1(0-0) + 0(6-0) + 3(9-0) = 27$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -\left[1(-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}\right]$$

$$= -[1(0-0) + 1(6-0) + 3(6-0)] = -24$$

$$C_{42} = (-1)^6 \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 1(0-0) + 1(9-0) + 0(6-0) = 9$$

$$|A| = 2C_{12} + 1C_{22} + 2C_{32} + 4C_{42}$$

$$= 2(0) + 1(27) + 2(-24) + 4(9) = 15$$

Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$.

$$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, \text{ then } |A| = 0, \text{ Hence A is singular}$$

$$A = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \text{ then } |A| \neq 0, \text{ Hence A is non-singular.}$$

Adjoint of a Matrix:

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is $C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$ ⇒

Hence $\text{adj } A = C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$

Inverse of a Matrix:

If A is a non-singular square matrix, then $A^{-1} = \frac{\text{adj } A}{|A|}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Example: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution: $|A| = 0 + 2(-2+3) - 3(-2+3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0,\quad A_{12} = 1,\quad A_{13} = 1$

$A_{21} = 2,\quad A_{22} = -3,\quad A_{23} = 2$

$A_{31} = 3,\quad A_{32} = -3,\quad A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution of Linear Equations by Matrices:

Consider the linear system:

$$a_{11}x_1 + a_{12}x_2 + \text{------} + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \text{------} + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \text{------} + a_{nn}x_n = b_n$$

It can be written as the matrix equation

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}}_{B} \rightarrow \boxed{AX = B} \rightarrow \boxed{X = A^{-1} B}$$

Example: Use matrices to find the solution set of

$$\begin{aligned} x + y - 2z &= 3 \\ 3x - y + z &= 5 \\ 3x + 3y - 6z &= 9 \end{aligned}$$

Solution:

Let $$A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$$

Since $$|A| = 3 + 21 - 24 = 0$$

Hence the solution of the given linear equations does not exists.

Example: Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$ $\quad |A| = -32 + 48 + 17 = 61$ $\quad$ So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

$X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix} = \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: $\{(x, y, z)\} = \{(-2, 0, 2)\}$

Homework

Use the matrices method to solve the following system of equations:

$$1.9\,P1 - 0.45\,P2 = 5125$$

$$-0.45\,P1 + 1.9\,P2 - 0.45\,P3 = 4000$$

$$-0.45\,P2 + 1.9\,P3 = 5800$$

THANK YOU