Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

Lecturer: Dr. Mohammed Idrees Al-Mossawy

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L9: Methods of Solving Systems of Linear Equations (Part 2)

Outlines

- Iterative Methods for Solving Linear Equations
- The Gauss-Seidel Method
- Strictly Diagonally Dominant Matrix
- Comparison of Iterative and Direct Methods for Solving Linear Equations
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Iterative Methods for Solving Linear Equations

- the idea in an iterative method is comprised of:
  - to make a first guess at the solution
  - to refine the guessed values in a stepwise manner to converge to the correct answer.

The Gauss-Seidel Method

The Gauss-Seidel method includes the following steps:

1. Make a first guess for the unknowns (*x2,x3,……,xn*).
2. Find the first unknown (*x1*).
3. Use *x1* with guessed values of (*x3, x4, ….,xn*) to find *x2*.
4. Use *x1* and the new value of *x2* with guessed values of (*x4,x5,…,xn*) to find *x3*.
5. Repeat the same procedure to find new value for the unknowns *x4,….,xn*.
6. Repeat steps #2 to #5 till the calculated values of the unknowns converge to the correct answer.

Example

Use the Gauss-Seidel iteration method to solve the following system of equations:

$$
\begin{aligned}
5x_1 - 2x_2 + 3x_3 &= -1 \\
-3x_1 + 9x_2 + x_3 &= 2 \\
2x_1 - x_2 - 7x_3 &= 3
\end{aligned}
$$

Solution:

To begin, write the system in the form:

$$
\begin{aligned}
x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\
x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\
x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2
\end{aligned}
$$

Use $(x_2, x_3) = (0, 0)$ as the initial approximation

Calculate x_1 : $x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$

Calculate x_2 : $x_2 = \frac{2}{9} + \frac{3}{9}(-0.200) - \frac{1}{9}(0) \approx 0.156$

Calculate x_3 : $x_3 = -\frac{3}{7} + \frac{2}{7}(-0.200) - \frac{1}{7}(0.156) \approx -0.508$

Continue iterations to reach the correct answer. Results are shown in the Table below.

n	0	1	2	3	4	5
x_1		−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

Note:

The iterative methods not always converges. That is, it is possible to apply an iterative method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, it is said that the method *diverges*.

Example: ***An Example of Divergence***

Results of applying the Gauss-Seidel method to the following system are shown in the Table below:

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}$$

n	0	1	2	3	4	5
x_1		-4	-174	-6124	$-214{,}374$	$-7{,}503{,}124$
x_2	0	-34	-1224	$-42{,}874$	$-1{,}500{,}624$	$-52{,}521{,}874$

- **With an initial approximation of $x_2 = 0$, the Gauss-Seidel method does not converge to the solution of the system of linear equations given in the above example.**
- **we will now look at a special type of coefficient matrix *A*, called a *strictly diagonally dominant matrix*, for which it is guaranteed that the Gauss-Seidel methods will converge.**

Strictly Diagonally Dominant Matrix

Definition

An $n \times n$ matrix A is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row. That is,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} \Rightarrow \begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| &> |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ &\vdots \\ |a_{nn}| &> |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|. \end{aligned}$$

Example

Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

(a)
$$\begin{aligned} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 + 2x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{aligned}$$

Solution

(a) The coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

is strictly diagonally dominant because $|3| > |-1|$ and $|5| > |2|$.

(b) The coefficient matrix

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \\ 3 & -5 & 1 \end{bmatrix}$$

is not strictly diagonally dominant because $a_{21} = 1,\ a_{22} = 0,\ a_{23} = 2,$
and it is not true that $|a_{22}| > |a_{21}| + |a_{23}|$.

- Interchanging the second and third rows in the original system of linear equations, however, produces the coefficient matrix

$$A' = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

and this matrix is strictly diagonally dominant

Theorem

If A is strictly diagonally dominant, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has a unique solution to which the Gauss-Seidel method will converge for any initial approximation.

Example: Interchanging Rows to Obtain Convergence

Interchange the rows of the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}$$

to obtain one with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.

Solution:

Begin by interchanging the two rows of the given system to obtain

$$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4. \end{aligned}$$

Then

$$\begin{aligned} x_1 &= \tfrac{6}{7} + \tfrac{1}{7}x_2 \\ x_2 &= \tfrac{4}{5} + \tfrac{1}{5}x_1 \end{aligned}$$

Starting with $x_2 = 0$, the results are

n	0	1	2	3	4	5
x_1		0.8571	0.9959	0.9999	1.000	1.000
x_2	0.0000	0.9714	0.9992	1.000	1.000	1.000

Note:

Do not conclude from the previous Theorem that strict diagonal dominance is a necessary condition for convergence of the Gauss-Seidel method.

Example:

The coefficient matrix of the system

$$\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

is not a strictly diagonally dominant matrix, and yet the **Gauss-Seidel** method converges to the solution when you use an initial approximation of $x_2 = 0$.

Comparison of Iterative and Direct Methods for Solving Linear Equations

1. Direct methods for solving a set of linear equations have algorithms that involve a fixed number of steps for a given size of problem.

2. Iterative methods start from a first guess at the solution and then applied a (usually simpler) algorithms to get better and better approximations to the true solution of the linear equations.

3. Usually the amount of "***work***" required for direct methods is smaller for smaller problems but iterative methods usually win out for larger problems.

Summary

- There are two methods to solve a set of linear equations; *direct methods* and *iterative methods.*

- For *direct methods*, the required *computation work is complex* but *its amount can be estimated*.

- For *iterative methods*, the required *computation work is simple* but *its amount cannot be estimated.*

Homework

Using the Gauss-Seidel Method, solve the following system of equations:

$$1.9\,P1 - 0.45\,P2 = 5125$$

$$-0.45\,P1 + 1.9\,P2 - 0.45\,P3 = 4000$$

$$-0.45\,P2 + 1.9\,P3 = 5800$$

THANK YOU