The Dot Product (Three dimension)

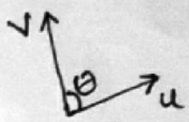
$u = \langle u_1, u_2, u_3 \rangle \quad , \quad v = \langle v_1, v_2, v_3 \rangle$

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

* The angle Between Two Vectors

$u = \langle u_1, u_2, u_3 \rangle \quad , \quad v = \langle v_1, v_2, v_3 \rangle$

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|u|\,|v|} \right)$$



Example 1

a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(3)$

$= -7$

b) $\left(\frac{1}{2} i + 3j + k\right) \cdot (4i - j + 2k) = \frac{1}{2}(4) + 3(-1) + (1)(2) = 1$

Example 2

Find the angle between $u = i - 2j - 2k$ and $v = 6i + 3j + 2k$

$u \cdot v = (1)(6) + (-2)(3) + (-2)(2) = -4$

$|u| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = 3$

$|v| = \sqrt{(6)^2 + (3)^2 + (2)^2} = 7$

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u|\,|v|} \right) = \cos^{-1}\left(\frac{-4}{(3)(7)} \right) \simeq 1.76 \text{ rad}$$

Orthogonal Vectors

~~Diff~~ **Definition** Vectors u and v are orthogonal if $u \cdot v = 0$

Example 3 $u = 3i - 2j + k$ is orthogonal to $v = 2j + 4k$ because

$$u \cdot v = (3)(0) + (-2)(2) + (1)(4) = 0$$

The vector projection of u onto v is the vector (three dimension)

$$\text{proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v$$

The scalar component of u in the direction of v is the scalar

$$|u| \cos\theta = \frac{u \cdot v}{|v|} = u \cdot \frac{v}{|v|}$$

Example 4 Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v

Solution

$$\text{proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \frac{(6)(1) + (3)(-2) + (2)(-2)}{\left(\sqrt{(1)^2 + (-2)^2 + (-2)^2}\right)^2}(i - 2j - 2k)$$

$$= -\frac{4}{9}(i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

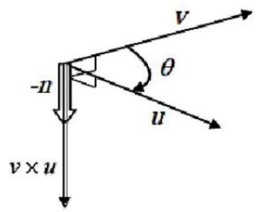
The scalar component of u in the direction of v

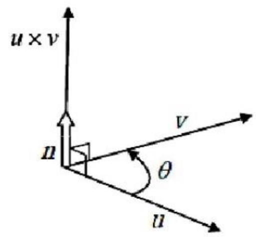
$$|u| \cos\theta = u \cdot \frac{v}{|v|} = (6i + 3j + 2k) \cdot \left(\frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k\right)$$

$$= 2 - 2 - \frac{4}{3} = \frac{-4}{3}$$

B. **The Cross Product :**

Two vector u and v in space if u and v are not parallel, they determine a plane, we select a unit vector n perpendicular to the plane by the right-hand rule. This means that we choose n to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from u to v





Then the cross product u x v is the vector defined as follows:

$$u \times v = (|u||v|\sin\theta)n$$

When we apply the definition to calculate the pair wise cross products of i , j, k we find:

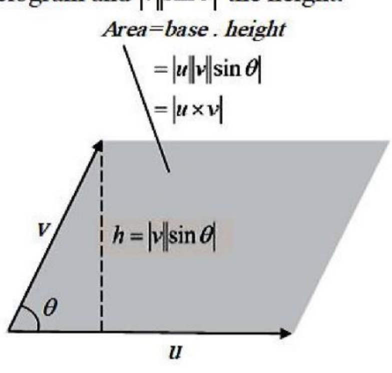
$$i \times j = -(j \times i) = k$$
$$j \times k = -(k \times j) = i$$
$$k \times i = -(i \times k) = j$$
$$i \times i = j \times j = k \times k = 0$$

$|u \times v|$ ***Is the area of a parallelogram***

Because n is a unit vector, the magnitude of $u \times v$ is:

$$|u \times v| = |u||v|\sin\theta|n| = |u||v|\sin\theta$$

This is the area of the parallelogram determined by ***u*** and ***v***, $|u|$ being the base of the parallelogram and $|v|\sin\theta|$ the height.



Properties of the cross product

if $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are any vectors and $\boldsymbol{r}$, $\boldsymbol{s}$ is are scalars, then:

1. $(ru)\times(sv)=(rs)(u\times v)$
2. $u\times(v+w)=u\times v+u\times w$
3. $v\times u=-(u\times v)$
4. $(v+w)\times u=v\times u+w\times u$
5. $0\times u=0$
6. $u\times(v\times w)=(u.w)v-(u.v)w$

EX: find u x v and v x u if u = 2i + j + k and v = -4i + 3 j + k ?

Sol:

$$u\times v=\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}i-\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}j+\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}k$$

$$=((1)(1)-(1)(3))i-((2)(1)-(1)(-4))j+((2)(3)-(1)(-4))k$$

$$=(1-3)i-(2+4)j+(6+4)k$$

$$=-2i-6j+10k$$

$$v\times u=-(u\times v)=2i+6j-10k$$

EXAMPLE 2 Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$

Solution The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ}=(2-1)\mathbf{i}+(1+1)\mathbf{j}+(-1-0)\mathbf{k}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

$$\overrightarrow{PR}=(-1-1)\mathbf{i}+(1+1)\mathbf{j}+(2-0)\mathbf{k}=-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}=\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$=6\mathbf{i}+6\mathbf{k}.$$

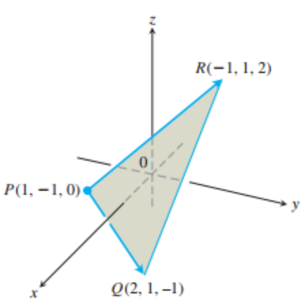


FIGURE The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\overrightarrow{PQ}\times\overrightarrow{PR}|$ (Example 3).

EXAMPLE 3 Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ}\times\overrightarrow{PR}|=|6\mathbf{i}+6\mathbf{k}|$$

$$=\sqrt{(6)^2+(6)^2}=\sqrt{2\cdot 36}=6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE 4 Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Solution

$$\mathbf{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{|\overrightarrow{PQ}\times\overrightarrow{PR}|}=\frac{6\mathbf{i}+6\mathbf{k}}{6\sqrt{2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}.$$ ■

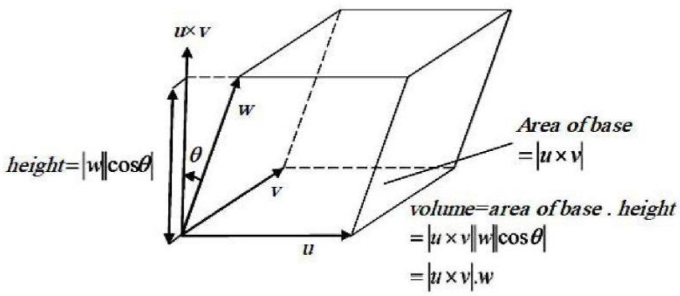
C. **Triple scalar or box product:**

The product (u x v).w is called the triple scalar product of u, v and w.

As you can see from the formula:

$$|(u \times v).w| = |u \times v||w||\cos\theta|$$

The absolute value of this product is the volume of the parallelepiped (parallelogram – side box) determined by ***u***, ***v*** and ***w***. the number $|u \times v|$ is the area of the base parallelogram. The number $|w||\cos\theta|$ is the parallelepiped's height. Because this geometry, $(u \times v).w$ is called **the box product** of ***u***,***v*** and ***w***.



To calculate the triple scalar product:

$$Let \quad u = a_1 i + b_1 j + c_1 k$$
$$v = a_2 i + b_2 j + c_2 k$$
$$w = a_3 i + b_3 j + c_3 k$$

Then

$$(u \times v).w = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

EX: Find volume of box (parallelepiped) determined by u = i + 2 j - k , v = -2i + 3k and w = 7 j - 4k ?

Sol:

$$(u \times v).w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix}(1) - \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix}(2) + \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}(-1)$$

$$= [(0)(-4) - (3)(7)](1) - [(-2)(-4) - (3)(0)](2) + [(-2)(7) - (0)(0)](-1)$$

$$= (0 - 21)(1) - (8 - 0)(2) + (-14 - 0)(-1)$$

$$= (-21)(1) - (8)(2) + (-14)(-1)$$

$$= -21 - 16 + 14 = \boxed{-23}$$

The volume is $|(u \times v).w| = 23$ units cubed