Al-Ayen University
College of Petroleum Engineering

Numerical Methods and Reservoir Simulation

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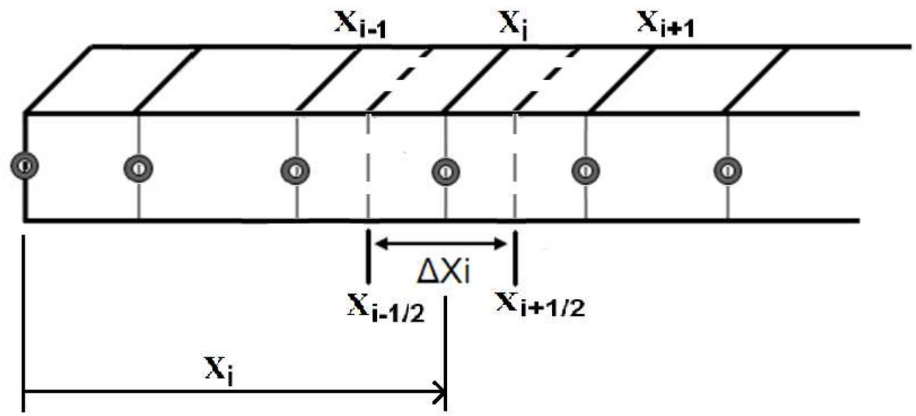
L16: Incorporation of Boundary Conditions/Point-Centered Grids

Outline

- Point-Centered Grids
- Incorporation of Boundary Conditions/Point-Centered Grids
  - Dirichlet Boundary Conditions
  - Neumann Boundary Conditions

Point-Centered Grids



Incorporation of Boundary Conditions/Point-Centered Grids

Dirichlet Boundary Conditions

• Dirichlet Boundary Conditions (pressure specified at the boundaries). For generality consider a heterogeneous reservoir with sources/sinks.

$$\text{PDE} \quad 1.127\times 10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

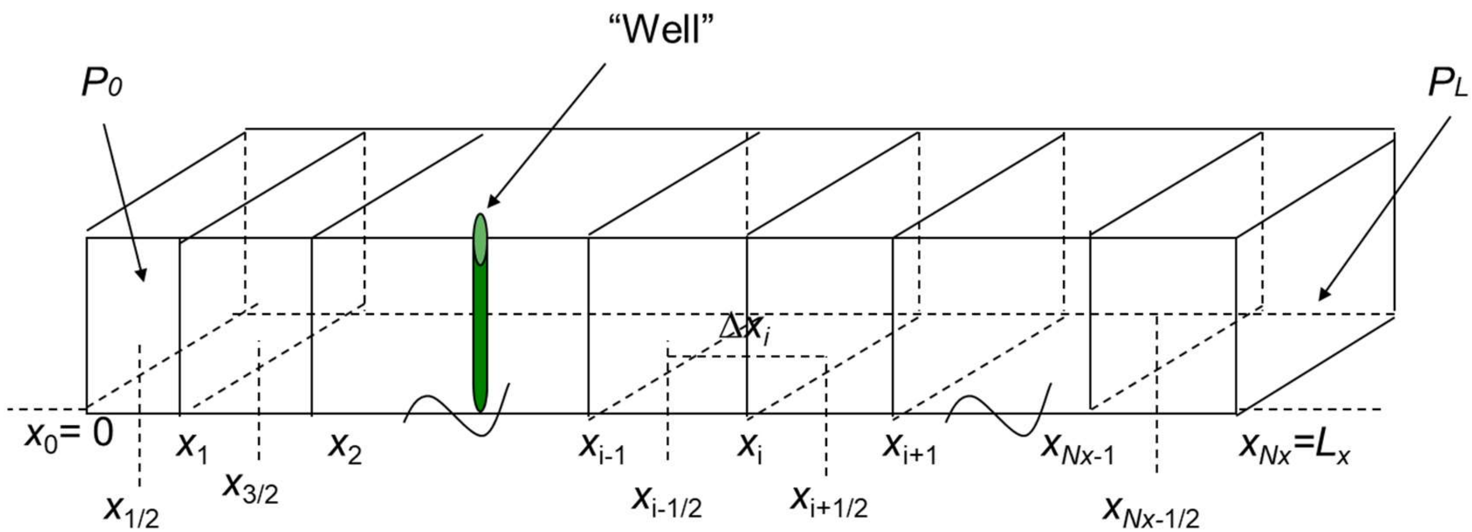
$$\text{IC} \quad p(x,0)=P_0,\ 0\le x\le L_x,$$

$$\text{BC} \quad p(x=0,t>0)=P_0$$

$$\text{BC} \quad p(x=L_x,t>0)=P_L$$

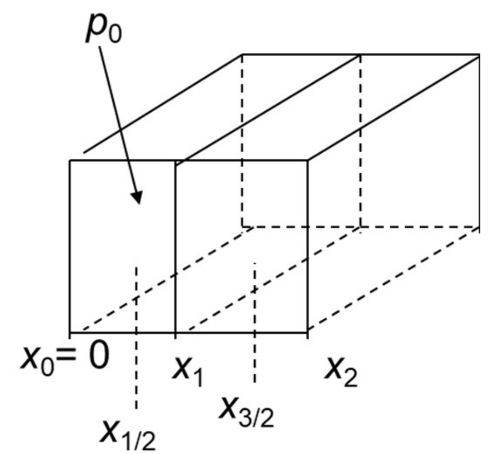
Point-Centered Grids

- Recall the grid system.



- General Implicit Difference Equations:

$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$



- For *i* =1,

$$-T_{x,1/2}p_0^{n+1}+\left(T_{x,3/2}+T_{x,1/2}+\widetilde{V}_1\right)p_1^{n+1}-T_{x,3/2}p_2^{n+1}=-q_{sc,1}^{n+1}B+\widetilde{V}_1 p_1^n$$

Because p_0 is known (BC), then for $i = 1$,

$$\left(T_{x,3/2}+T_{x,1/2}+\widetilde{V}_1\right)p_1^{n+1}-T_{x,3/2}p_2^{n+1}=-q_{sc,1}^{n+1}B+\widetilde{V}_1 p_1^n+T_{x,1/2}p_0^{n+1}$$

- For i = 2,3,…,N_x-2

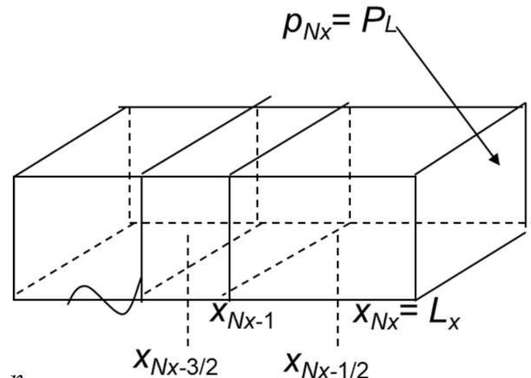
$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$

where we evaluate *transmissibilities* from:

$$T_{x,i+1/2}=1.127\times10^{-3}\frac{\lambda_{x,i+1/2}wh}{(x_{i+1}-x_i)}\ and\ T_{x,i-1/2}=1.127\times10^{-3}\frac{\lambda_{x,i-1/2}wh}{(x_i-x_{i-1})}$$

- General Implicit Difference Equation:

$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$



- For $i=N_x$-1

$$-T_{x,N_x-3/2}p_{N_x-2}^{n+1}+\left(T_{x,N_x-1/2}+T_{x,N_x-3/2}+\widetilde{V}_{N_x-1}\right)p_{N_x-1}^{n+1}-T_{x,N_x-1/2}p_{N_x}^{n+1}=-q_{sc,N_x-1}^{n+1}B+\widetilde{V}_{N_x-1}p_{N_x-1}^n$$

- Note that N_x is the total number of Grid points including the grid points at the left- and right-hand reservoir Boundaries.

- Because p_{Nx} is known (from BC), then for $i = N_{x-1}$,

$$-T_{x,Nx-3/2}p_{Nx-2}^{n+1}+\left(T_{x,Nx-3/2}+T_{x,Nx-1/2}+\widetilde{V}_{Nx-1}\right)p_{Nx-1}^{n+1}=-q_{sc,Nx-1}^{n+1}B+\widetilde{V}_{Nx-1}p_{Nx-1}^n+T_{x,Nx-1/2}p_{Nx}^{n+1}$$

Note: We will have N_x-1 unknowns and N_x-1 equations to solve here.

Incorporation of Boundary Conditions/Point-Centered Grids

Neumann Boundary Conditions

- Neumann <u>Boundary Conditions</u> (pressure gradients specified at the boundaries).
- For generality consider a heterogeneous reservoir with sources/sinks.

$$\text{PDE} \quad 1.127\times10^{-3}\frac{\partial}{\partial x}\left(\frac{k_x}{\mu}\frac{\partial p}{\partial x}\right)-\frac{q_{sc}(x,t)B}{V_b}=\frac{\phi c_t}{5.615}\frac{\partial p}{\partial t},\ 0<x<L_x,\ t>0$$

$$\text{IC} \quad p(x,t=0)=P_{Initial},\ 0\le x\le L_x,$$

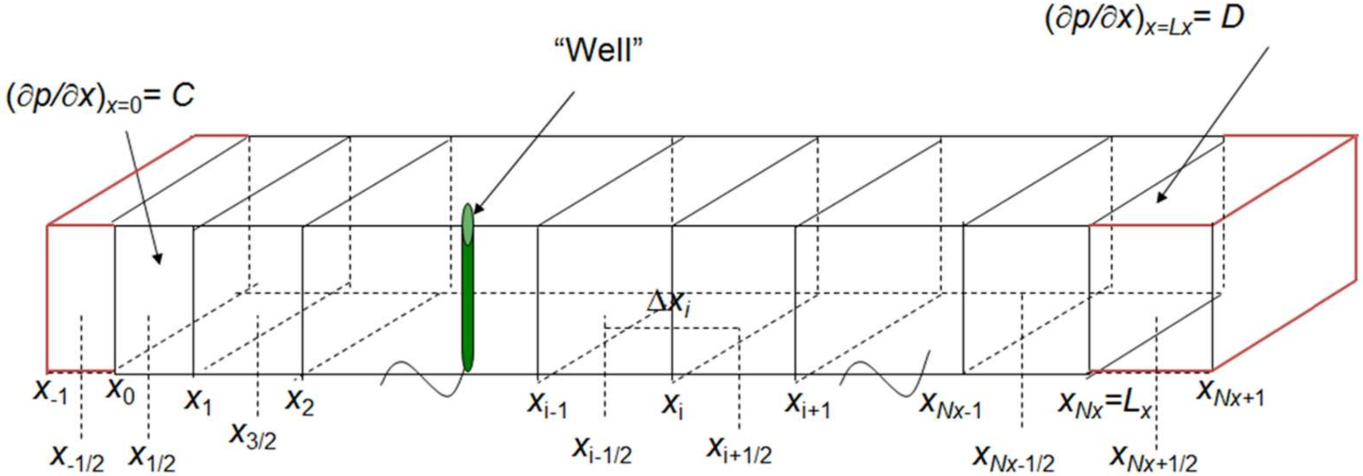
$$\text{BC} \quad \frac{\partial p(x=0,t>0)}{\partial x}=C$$

$$\text{BC} \quad \frac{\partial p(x=L_x,t>0)}{\partial x}=D$$

Note: if $C=0$ and/or $D=0$, then we have no-flow (closed) boundary.

Point-Centered Grids

- Recall the grid system.



Incorporating Neumann BCs to a point-centered grid as shown above can be handled by reflections (images) of the first and last grids (red colored in above figure). Note that if a source/sink at these blocks, then imaging will also duplicate the source/sink.

- General Implicit Difference Equation: Multiply both sides by the bulk volume of the grid-block *i*, $V_{b,i} = \Delta x_i wh$

$$1.127\times10^{-3}\Delta x_i wh\left[\frac{\lambda_{x,i+1/2}\left(\frac{p_{i+1}^{n+1}-p_i^{n+1}}{x_{i+1}-x_i}\right)-\lambda_{x,i-1/2}\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{x_i-x_{i-1}}\right)}{x_{i+1/2}-x_{i-1/2}}\right]-q_{sc,i}^{\;n+1}B=\frac{(\phi c_t)_i\Delta x_i wh}{5.615}\left(\frac{p_i^{n+1}-p_i^n}{\Delta t^{n+1}}\right)$$

$$T_{x,i+1/2}\left(p_{i+1}^{n+1}-p_i^{n+1}\right)-T_{x,i-1/2}\left(p_i^{n+1}-p_{i-1}^{n+1}\right)-q_{sc,i}^{\;n+1}B=\widetilde{V}_i\left(p_i^{n+1}-p_i^n\right)$$

or

$$-T_{x,i-1/2}p_{i-1}^{n+1}+\left(T_{x,i+1/2}+T_{x,i-1/2}+\widetilde{V}_i\right)p_i^{n+1}-T_{x,i+1/2}p_{i+1}^{n+1}=-q_{sc,i}^{n+1}B+\widetilde{V}_i p_i^n$$

$$T_{x,i+1/2}=1.127\times10^{-3}\frac{\lambda_{x,i+1/2}wh}{\left(x_{i+1}-x_i\right)},\ T_{x,i-1/2}=1.127\times10^{-3}\frac{\lambda_{x,i-1/2}wh}{\left(x_i-x_{i-1}\right)},\ and\ \ \widetilde{V}_i=\frac{(\phi c_t)\Delta x_i wh}{5.615\Delta t^{n+1}}$$

Difference equation is the same as before!

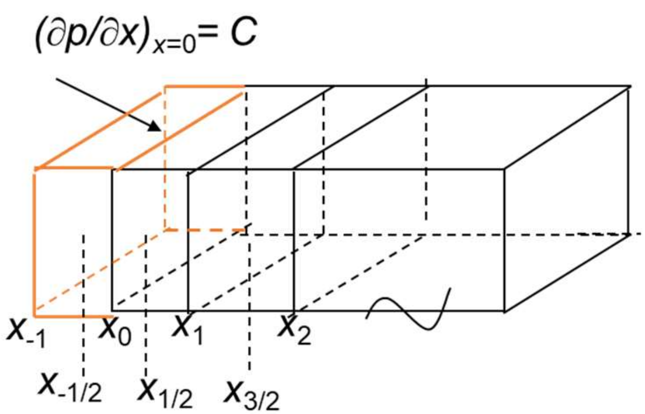
- The General Implicit Difference Equations for i=1,2, …,N_{x-1}.

$$T_{x,i+1/2}\left(p_{i+1}^{n+1}-p_i^{n+1}\right)-T_{x,i-1/2}\left(p_i^{n+1}-p_{i-1}^{n+1}\right)-q_{sc,i}^{\ n+1}B=\widetilde{V}_i\left(p_i^{n+1}-p_i^{n}\right)$$

  - For i = 0, we have

$$T_{x,1/2}\left(p_1^{n+1}-p_0^{n+1}\right)-T_{x,-1/2}\left(p_0^{n+1}-p_{-1}^{n+1}\right)-2q_{sc,0}^{\ n+1}B=\widetilde{V}_0\left(p_0^{n+1}-p_0^{n}\right)$$

Note that $T_{x,1/2}=T_{x,-1/2}$ because of imaging



p_0 and p_{-1} are the pressures at x_0, and x_{-1}. This adds two additional unknowns to the unknown pressures at the interior grid-points; p_i, i=1,2,…,N_x-1.

- Recall that boundary condition at $x = 0$ can be approximated as:

$$\left(\frac{\partial p}{\partial x}\right)_{x_0=0}^{n+1} = \frac{p_1^{n+1} - p_{-1}^{n+1}}{x_1 - x_{-1}} = C$$

$$p_1^{n+1} = p_{-1}^{n+1} + C(x_1 - x_{-1})$$

If $C = 0$ (no-flow or closed left-hand boundary), then

$$p_1^{n+1} = p_{-1}^{n+1}$$

$$T_{x,1/2}\left(p_1^{n+1} - p_0^{n+1}\right) - T_{x,-1/2}\left(p_0^{n+1} - \cancelto{p_1}{p_{-1}^{n+1}}\right) - 2q_{sc,0}^{\ \ n+1}B = \tilde{V}_0\left(p_0^{n+1} - p_0^{n}\right)$$

$$2T_{x,1/2}\left(p_1^{n+1} - p_0^{n+1}\right) - 2q_{sc,0}^{\ \ n+1}B = \tilde{V}_0\left(p_0^{n+1} - p_0^{n}\right)$$

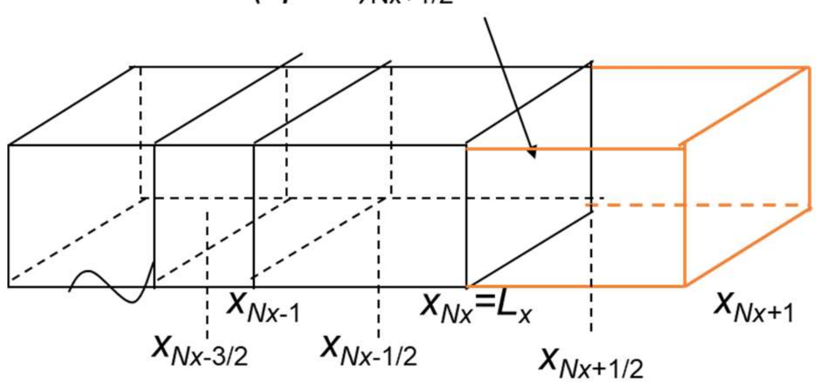
- The General Implicit Difference Equations for i=1,2, …,N_{x-1}.

$$T_{x,i+1/2}\left(p_{i+1}^{n+1}-p_{i}^{n+1}\right)-T_{x,i-1/2}\left(p_{i}^{n+1}-p_{i-1}^{n+1}\right)-q_{sc,i}^{\;n+1}B=\widetilde{V}_{i}\left(p_{i}^{n+1}-p_{i}^{n}\right)$$

- For $i = N_x$,

$$T_{x,Nx+1/2}\left(p_{Nx+1}^{n+1}-p_{Nx}^{n+1}\right)-T_{x,Nx-1/2}\left(p_{Nx}^{n+1}-p_{Nx-1}^{n+1}\right)-2q_{sc,Nx}^{\;n+1}B=\widetilde{V}_{Nx}\left(p_{Nx}^{n+1}-p_{Nx}^{n}\right)$$

Note that $T_{x,Nx+1/2}=T_{x,Nx-1/2}$ because of imaging



p_{Nx} and p_{Nx+1} are the pressures at x_{Nx}, and x_{Nx+1}. These are two more additional unknowns.

- Recall that boundary condition at $x_{Nx} = L_x$ can be approximated as:

$$\left(\frac{\partial p}{\partial x}\right)^{n+1}_{x_{Nx}=0} = \frac{p^{n+1}_{N_{x+1}} - p^{n+1}_{N_x-1}}{x_{N_x+1} - x_{N_x-1}} = D$$

$$p^{n+1}_{N_x+1} = p^{n+1}_{N_x-1} + D(x_{N_x+1} - x_{N_x-1})$$

If D = 0 (no-flow or closed right-hand boundary), then

$$p^{n+1}_{N_x+1} = p^{n+1}_{N_x-1}$$

p_{Nx-1}

$$T_{x,Nx+1/2}\left(p^{n+1}_{Nx+1} - p^{n+1}_{Nx}\right) - T_{x,Nx-1/2}\left(p^{n+1}_{Nx} - p^{n+1}_{Nx-1}\right) - 2q_{sc,Nx}^{\ n+1}B = \widetilde{V}_{Nx}\left(p^{n+1}_{Nx} - p^{n}_{Nx}\right)$$

$$-2T_{x,N_x-1/2}\left(p^{n+1}_{Nx} - p^{n+1}_{Nx-1}\right) - 2q_{sc,Nx}^{\ n+1}B = \widetilde{V}_{Nx}\left(p^{n+1}_{Nx} - p^{n}_{Nx}\right)$$

- In summary, we have the following system of equations to solve:

$$T_{x,1/2}\left(p_1^{n+1} - p_0^{n+1}\right) - T_{x,-1/2}\left(p_0^{n+1} - p_{-1}^{n+1}\right) - 2q_{sc,0}^{\ \ n+1}B = \widetilde{V}_0\left(p_0^{n+1} - p_0^{n}\right)$$

$$p_{-1}^{n+1} = p_1^{n+1} - C(x_1 - x_{-1})$$

$$T_{x,i+1/2}\left(p_{i+1}^{n+1} - p_i^{n+1}\right) - T_{x,i-1/2}\left(p_i^{n+1} - p_{i-1}^{n+1}\right) - q_{sc,i}^{\ \ n+1}B = \widetilde{V}_i\left(p_i^{n+1} - p_i^{n}\right)$$

for i=1,2,…,N_x-1

$$T_{x,Nx+1/2}\left(p_{Nx+1}^{n+1} - p_{Nx}^{n+1}\right) - T_{x,Nx-1/2}\left(p_{Nx}^{n+1} - p_{Nx-1}^{n+1}\right) - 2q_{sc,Nx}^{\ \ n+1}B = \widetilde{V}_{Nx}\left(p_{Nx}^{n+1} - p_{Nx}^{n}\right)$$

$$p_{N_x+1}^{n+1} = p_{N_x-1}^{n+1} + D(x_{N_x+1} - x_{N_x-1})$$

N_x+1 unknowns and N_x+1 equations if we combine the first two and the last two equations. Unknowns are p_0, p_1,…,p_{Nx}. Note that we solve the pressures at the boundary as well.

THANK YOU